令和５年度　博士論文

A Novel Method of Diagonal-Inner Outer Factorization

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Chapter 1

Introduction

In the realm of computational mathematics, factorization stands as a cornerstone, underpinning numerous applications across various disciplines. From solving complex matrix equations to optimizing computational algorithms, the ability to decompose mathematical entities efficiently and accurately is indispensable. At the heart of this lies a particular focus on matrix factorization, a process that unravels the complexities of matrices into more manageable and interpretable forms.

1.1 Significance of Diagonal-Inner Outer Factorization

Within the spectrum of matrix factorization methods, diagonal-inner outer factorization emerges as a significant and nuanced area. This method involves the decomposition of a matrix into 'inner' and 'outer' components, offering a unique approach to understanding and manipulating matrix structures. Specifically, the focus on the diagonal structure in the inner part of the factorization presents intriguing possibilities for enhanced computational efficiency and deeper mathematical insights.

The relevance of diagonal-inner outer factorization extends beyond theoretical interest; it has practical implications in fields such as signal processing, control theory, and numerical linear algebra. In these domains, the method's ability to provide clearer insights into matrix behaviors and improve computational performances is invaluable. As we delve into this thesis, the goal is to explore and expand upon the existing methodologies of diagonal-inner outer factorization, with a particular emphasis on the utilization of a diagonal function as the inner component. This approach is not only a testament to the evolving landscape of mathematical factorization but also an endeavor to contribute meaningfully to this vital area of computational mathematics.

The exploration of diagonal-inner outer factorization methods in mathematical research has led to various applications and theoretical developments. While traditional approaches have provided foundational insights, they often encompass limitations in terms of computational efficiency and adaptability to complex matrix structures. These methodologies, primarily designed for general purposes, sometimes fall short in addressing specific nuances, especially when dealing with matrices that exhibit uniqueness.

One of the principal challenges in current factorization techniques is their handling of the inner matrix component. Typically, these methods do not prioritize the specific structure of the inner matrix, which can be crucial for certain mathematical and computational applications. This oversight can lead to less efficient computations, especially in scenarios where the properties of the inner matrix significantly influence the overall factorization process. Furthermore, existing techniques may not fully leverage the potential advantages of tailoring the inner component to suit specific matrix types, leading to missed opportunities for optimization.

In light of these challenges, there emerges a clear gap in the research: the need for a method that explicitly considers the structure of the inner matrix component in diagonal-inner outer factorization. This thesis proposes bridging this gap by introducing a novel approach where the inner function is a diagonal function. This method is anticipated to offer several advantages, including enhanced computational efficiency, greater adaptability to various matrix structures, and potentially more insightful mathematical interpretations. By focusing on a diagonal function as the inner part of the factorization, the research aims to address the limitations of existing methods and open up new avenues for application in computational mathematics and related fields.

1.2 Proposing a Novel Diagonal-Inner Outer Factorization Method

The primary aim of this thesis is to propose and explore an innovative approach to diagonal-inner outer factorization, where the focus is placed on employing a diagonal function as the inner component. This approach is not just a mere variation of existing techniques, but a fundamental rethinking of how diagonal inner outer factorization can be conducted more effectively and efficiently. First, The development of a Diagonal Function-Based inner outer Factorization Method. The cornerstone of this thesis is to develop a factorization method that leverages the unique advantages of a diagonal inner function. This method is expected to optimize the factorization process, particularly for matrices where the diagonal structure of the inner component can play a crucial role. Second, Computational Efficiency. A key objective is to demonstrate that this new method can lead to significant improvements in computational efficiency. This includes reducing computational time and resources, especially in scenarios involving large or complex matrices. Third, Theoretical and Practical Implications. The research aims to explore the theoretical underpinnings of this approach and its implications for practical applications in areas such as numerical analysis, engineering, and data processing.

By achieving these objectives, this research seeks to fill the identified gap in the field of diagonal-inner outer factorization. The introduction of a diagonal function as the inner component represents a significant shift in how these factorizations are conceptualized and implemented. Ultimately, this research aims to contribute not only to the body of mathematical knowledge but also to the practical methodologies employed in computational mathematics and its related disciplines.

1.3 History in Inner Outer Factorization

In recent years, the field of inner outer factorization has witnessed significant advancements, marked by innovative methodologies and enhanced computational techniques. These developments have not only refined the existing approaches but have also introduced new perspectives and solutions to longstanding challenges in matrix factorization. The work of Hara and Sugie proposed a simplified state-space procedure for inner outer factorization, offering a unique solution to the challenges posed by traditional methods [1]. This approach simplified the problem and provided a unique solution, indicating the potential for more streamlined and efficient factorization techniques. A groundbreaking advancement came with Varga's introduction of an efficient recursive zeros dislocation technique for the inner outer factorization of rational transfer matrices [2] . This method marked a significant step forward in handling complex matrices, showcasing improved accuracy and efficiency in factorization processes. Another notable development was presented by Xia, who proposed an inner outer preconditioner for symmetric positive definite matrices based on HSS matrix representation [3]. This approach combined the advantages of direct and triangular HSS methods, leading to a solution that was both scalable and robust, especially for large-scale computational problems. The recent work by Liao and Zheng introduced iterative inner/outer approximations for scalable semidefinite programs using block factor-width-two matrices [4]. This innovative method offered a flexible balance between numerical efficiency and solution quality, demonstrating the potential for faster and more reliable factorization in complex mathematical programming.

1.4 Factorization Techniques Comparison

Inner outer factorization is specialized for stability and control analysis in matrices, the other methods (LU, Cholesky, QR, SVD) offer broader applications in linear algebra and are chosen based on the specific requirements of the matrix and the problem at hand.

1.5 Development Guidelines in New Diagonal-Inner Outer Factorization

The journey of diagonal-inner outer factorization begins with the broader history of matrix factorization in mathematics. Originating from the fundamental need to simplify and solve matrix equations, factorization methods have evolved significantly over the years. Initially, these techniques were developed to address linear algebraic problems, with early mathematicians laying the groundwork through concepts like LU factorization [5] and Cholesky factorization [6]. diagonal-inner outer factorization emerged as a distinct method within this field, focusing on decomposing a matrix into inner and outer components. This approach was notably different from traditional factorization methods, as it specifically considered the structure and properties of the inner matrix. The significance of this method became more pronounced with the increasing complexity of mathematical problems and the advent of high-speed computing.

The development of diagonal-inner outer factorization can be attributed to several key contributions in the field. Notably, Strang ' s work provided a comprehensive understanding of how outer products could be used to produce rows and columns of a matrix, highlighting the importance of elimination, echelon form, and Gram-Schmidt processes in matrix factorization [7]. These foundational methods played a critical role in the development of diagonal-inner outer factorization techniques.

Table 1.1: Comparison of inner outer factorization method with other method

Factorization Method	Purpose	Characteristics	Applications	Computational Efficiency
LU Decomposition	Decomposes a matrix into lower and upper triangular matrices	Useful for solving linear equations, inversion, determinant	Numerical analysis, linear equations	General-purpose, efficient for linear systems
Cholesky Factorization	Special case of LU for Hermitian, positive definite matrices	Lower triangular and its conjugate transpose	Positive definite matrix problems in physics, engineering	Efficient for specific matrix types
QR Factorization	Decomposes a matrix into an orthogonal and an upper triangular matrix	Orthogonal matrix Q and upper triangular matrix R	Linear algebra, least squares problems	Efficient in least squares calculations
Singular Value Decomposition (SVD)	Decomposes a matrix into eigenvectors and eigenvalues matrices	Three matrices showing geometric and algebraic properties	Signal processing, data compression, PCA	Suitable for data-related computation
Inner-Outer Factorization	Separates a matrix into stable and unstable dynamics	Inner part often unitary; outer part deals with gain	Control theory, signal processing	Efficient in systems analysis contexts

1.6 Concluding the Introduction

As we conclude this introduction, it's crucial to reiterate the significance of this research in the broader context of diagonal-inner outer factorization. This research is poised to make meaningful contributions to the field of computational mathematics. It is expected that the development and exploration of this novel factorization method will not only address the current limitations identified in existing techniques but also pave the way for more innovative and efficient approaches in the future. The implications of this work extend beyond theoretical advancements, potentially influencing practical applications in various scientific and engineering domains.

Chapter 2

A NOVEL METHOD OF DIAGONAL-INNER OUTER FACTORIZATION

2.1 Introduction

In numerous control problems, the inner outer factorization serves as a mathematical procedure that partitions a matrix or operator into two distinctive elements: an inner function and an outer function. This strategic technique is employed to simplify the architecture of a matrix or operator, further promoting efficient numerical computations. inner outer factorization finds widespread utilization across several fields, including but not limited to numerical linear algebra, optimization, and signal processing. This method also plays a significant role in various other mathematical and engineering domains. Notable instances of inner outer factorizations comprise the Cholesky factorization and QR factorization. QR factorization given matrix is factored into the product of two matrices: a unitary or orthogonal matrix (Q) and an upper triangular matrix (R) [8]. Given its utility, the inner outer factorization method for stable transfer function matrices has proven to be an efficacious instrument for the analysis and synthesis of robust controllers, as well as the processing and communication of data [9, 10, 11, 12]. This technique is renowned for computer computations based on the state-space representation [11, 12, 13, 15, 2, 16, 17, 18].

The inner outer factorization serves as a computational paradigm in the context of H_2 and H_∞ control systems, offering the potential to fragment a stable and proper system $G(s)$ into inner and outer functions [9, 10, 11, 12, 15]. In a related development, Dewild and Veen devised an approach for the inversion of infinite systems of linear equations which are represented by a discrete time-varying dynamical system. Their method extrapolates on the classical matrix inversion theory by deploying inner outer factorizations, which operate analogously to QR-factorization in linear algebra. The algorithms they developed to this effect, coined 'square root' algorithms, not only avoid the requirement for multiple eigenvalue determination but are also efficient and linear in the volume of data [19].In a separate study, Gu scrutinized inner outer factorization for strictly proper transfer matrices, providing characterizations to the solution of this specific factorization problem and devising a computational algorithm for its resolution [20]. Boche and Pohl pioneered a series of algorithms to perform discrete inner outer, coprime, and spectral factorizations directly on the state-space realizations of discrete systems. Their approach circumvents the standard bilinear transformation, emphasizing the discrete algebraic Riccati equation and state-space realizations for factorizations [21]. Kase and Mutoh proposed an innovative and computationally efficient methodology for factorization that takes into account the specific attributes of the inner and outer components. Employing a recursive zeros dislocation technique, their approach manages generalized Lyapunov equations and is applicable regardless of whether G is proper or of full column/row rank, expanding the applicability of existing techniques to arbitrary rational matrices while avoiding the use of Riccati equations [22]. In a unique take on the matter, Helmer introduced an inner outer factorization within non-commutative Hardy algebras $H^\infty(E)$. His approach encapsulates various algebraic structures, centralizing around the Hardy algebra. It harnesses a general version of the Wold decomposition and factors a vector within the underlying Hilbert space, followed by an element of the algebra's commutant, using duality concepts for W^*-correspondences to foster new factorization approaches within this algebraic framework [23]. Reis and Vigt extended a method to non-square transfer function matrices, contemplating the derivation of the right interactor for the inner outer factorization. The proposed interactor showcases all-pass properties in discrete time, with all its zeros located at the origin. However, the authors addressed the zeros assignment of the interaction, considering the instability of the origin in continuous-time systems [24]. Frazho and Ran, in their note, utilized operator methods to address a rational inner outer factorization problem for wide functions. They engaged

Wiener-Hopf operators, Hankel operators, and invariant subspaces for the backward shift, hoping to yield significant insight into the inner outer factorization problem [25]. Recent studies can be used to improve the method, Huang, Wang, and Ramirez-Mendoza examine optimal controllers for model predictive control, focusing on theoretical analysis and practical application. Theoretically, it employs variation analysis and linear matrix inequality for controller design and stability analysis. Practically, it validates the theories using two simulation examples, applying an online subgradient descent algorithm for constrained optimization problems [26].

Given the considerable significance and wide-ranging applications of the inner outer factorization problem, numerous techniques have been explored extensively [9, 10, 11, 12, 15, 16, 2, 14, 17, 18, 21, 22, 24]. In these methods, the inner function is typically not a diagonal function. Despite the ability to control many systems even when the inner function is non-diagonal, the control problem may become more challenging under such conditions. Conversely, systems become more tractable in terms of control when the inner function is a diagonal function. Moreover, the diagonal function can streamline the analysis and design of control algorithms. However, the examination of a diagonal-inner outer factorization method remains absent in the literature.

In this thesis, we propose a novel method of diagonal-inner outer factorization. Our proposed methodology elucidates an inner function through the use of a straightforward diagonal function. The research exhibits interest in applying bi-proper transfer function and strictly proper transfer function for factorization. The structure of this thesis has been thoughtfully divided into two distinctive parts for comprehensive understanding.

In the initial segment, we offer a detailed explanation regarding the problem formulation for bi-proper systems in section 2.2. Our proposed methodology introduces a unique way of executing diagonal-inner outer factorization for bi-proper systems, which integrates the idea of the inverse system and inner transformation in section 2.3. The effectiveness of this proposed method is vividly demonstrated through a variety of numerical examples illustrated in Section 2.7.1. In the second segment, our focus is concentrated on the problem formulation for strictly proper systems in section 2.4. Consequently, the innovative method of diagonal-inner outer factorization for strictly proper systems is introduced 2.5. We present several numerical examples that effectively illustrate the method, and these examples can be found in Section 2.7.2. Section 2.8 gives concluding remarks.

2.2 Problem formulation for bi-proper systems

Consider a linear time-invariant systems of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right. , \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^m$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. It is assumed that all eigenvalues of A are in the open left half plane, (A, B) is stabilizable, (C, A) is detectable and

$$\text{rank } D = m. \tag{2.2}$$

In addition, it is assumed that the system in (2.1) has no zero on the imaginary axis, that is,

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = m \ \ \forall s = j\omega(-\infty < \omega < \infty). \tag{2.3}$$

It is assumed that unstable zeros of the system in (2.1) are row unstable zeros. That is, all unstable zeros z of $G(s)$ are located as

$$\text{rank} \begin{bmatrix} A - zI & B \\ C_i & D_i \end{bmatrix} < m(i = 1, \dots, m), \tag{2.4}$$

where

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \tag{2.5}$$

and

$$D = \begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix}. \tag{2.6}$$

The transfer function from $u(s)$ to $y(s)$ in (2.1) is denoted by

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B+D \\ &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{RH}_\infty^{m\times m}(s). \end{aligned} \tag{2.7}$$

The problem considered in this thesis is to factorize $G(s)$ in (2.7) as

$$G(s) = G_i(s)G_o(s), \tag{2.8}$$

where $G_i(s) \in \mathbb{R}^{m\times m}(s)$ is an diagonal-inner function and $G_o(s) \in \mathbb{R}^{m\times m}(s)$ is an outer function.

Having elaborated on the intricate aspects of problem formulation for bi-proper systems, we are now equipped with a comprehensive understanding of the challenges and opportunities inherent in these systems. The complexities and specificities outlined in this section serve as the foundational premise for our proposed solutions.

Moving into Section 3, we will introduce the idea of diagonal-inner outer factorization, a pioneering methodology tailored to address the nuanced challenges identified in the bi-proper systems. The diagonal-inner outer factorization emerges as a pragmatic approach, bridging the theoretical constructs and practical implementations, effectively managing the problematics elucidated herein.

2.3 Diagonal-inner outer factorization for bi-proper systems

In this section, an idea for a method of diagonal-inner outer factorization for $G(s)$ is explained.

This method adapts the combination of inverse system and inner transformation. From (2.2), there exists

$$\begin{aligned} \bar{G}(s) &= G^{-1}(s) \\ &= \left[\begin{array}{c|c} A-BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \\ &\equiv \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]. \end{aligned} \tag{2.9}$$

$\bar{G}(s)$ can be factorized as

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array}\right], \tag{2.10}$$

where $\bar{G}_i(s) \in \mathbb{RH}_\infty^m (i=1,\dots,m)$. Minimum realization of $\bar{G}_i(s)(i=1,\dots,m)$ in (2.10) is denoted by

$$\bar{G}_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \quad (i=1,\dots,m), \tag{2.11}$$

where $\bar{A}_i \in \mathbb{R}^{n_i\times n_i}(i=1,\dots,m)$, $\bar{B}_i \in \mathbb{R}^{n_i\times 1}(i=1,\dots,m)$, $\bar{C}_i \in \mathbb{R}^{m_i\times n_i}(i=1,\dots,m)$ and $\bar{D}_i \in \mathbb{R}^{m_i\times 1}(i=1,\dots,m)$. Using $K_i \in \mathbb{R}^{1\times n_i}(i=1,\dots,m)$ to make $\bar{A}_i - \bar{B}_iK_i$ have no eigenvalue in the closed right half plane, we have

$$\begin{aligned} \left[\begin{array}{c|c} \bar{A}_i-\bar{B}_iK_i & \bar{B}_i \\ \hline \bar{C}_i-\bar{D}_iK_i & \bar{D}_i \end{array}\right] &= \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array}\right]^{-1} \\ &= \bar{G}_i(s)G_{Ki}(s) \;\; (i=1,\dots,m), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} G_{Ki}(s) &= \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|c} \bar{A}_i-\bar{B}_iK_i & \bar{B}_i \\ \hline -K_i & 1 \end{array}\right] \quad (i=1,\dots,m). \end{aligned} \tag{2.13}$$

This yields

$$\hat{G}(s) = \bar{G}(s)G_K(s), \tag{2.14}$$

where

$$\begin{aligned}\hat{G}(s) \\ = \left[\begin{array}{ccc} \bar{C}_1\left(sI-\bar{A}_1+\bar{B}_1K_1\right)^{-1}\bar{B}_1+\bar{D}_1 & \dots & \bar{C}_m\left(sI-\bar{A}_m+\bar{B}_mK_m\right)^{-1}\bar{B}_m+\bar{D}_m \end{array}\right],\end{aligned} \tag{2.15}$$

$$\begin{aligned} G_K(s) &= \operatorname{diag}\left[\begin{array}{ccc} \left\{1+K_1\left(sI-\bar{A}_1\right)^{-1}\bar{B}_1\right\}^{-1} & \dots & \left\{1+K_m\left(sI-\bar{A}_m\right)^{-1}\bar{B}_m\right\}^{-1} \end{array}\right] \\ &= \left[\begin{array}{ccc|ccc} \bar{A}_1-\bar{B}_1K_1 & & 0 & \bar{B}_1 & & \\ & \ddots & & & \ddots & \\ 0 & & \bar{A}_m-\bar{B}_mK_m & 0 & & \bar{B}_m \\ \hline -K_1 & & 0 & 1 & & \\ & \ddots & & & \ddots & \\ 0 & & -K_m & 0 & & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \end{aligned} \tag{2.16}$$

and

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array}\right]. \tag{2.17}$$

From (2.9) and (2.14), we have

$$G(s) = G_K(s)\hat{G}^{-1}(s). \tag{2.18}$$

From (2.15), $\hat{G}(s)$ is stable and of minimum phase. If $K_i(i = 1, \dots, m)$ are settled to make $G_K(s)$ inner function, $G(s)$ is factorized as

$$G(s) = G_i(s)G_o(s), \tag{2.19}$$

where $G_i(s)$ is an inner function and given by

$$G_i(s) = G_K(s) \tag{2.20}$$

and $G_o(s)$ is an outer function and given by

$$G_o(s) = G_i^{-1}(s)G(s) = \hat{G}^{-1}(s). \tag{2.21}$$

It is obvious that $G_K(s)$ in (2.16) is an diagonal-inner function. In addition, from the assumption that all unstable zeros of $G(s)$ satisfy (2.4) and (2.21), $G_o(s)$ is obviously stable and of minimum-phase.

The rest is to obtain $K_i(i = 1, \dots, m)$ to make $G_K(s)$ an inner function. On a design method for $K_i(i = 1, \dots, m)$, we have following theorem.

Theorem 2.3.1 *If $K_i(i = 1, \dots, m)$ are settled by*

$$K_i = \bar{B}_i^T X_i \ \ (i = 1, \dots, m), \tag{2.22}$$

where $X_i \geq 0(j = 1, \dots, m)$ is the unique solution of the Riccati equation

$$X_i\bar{A}_i + \bar{A}_i^T X_i - X_i\bar{B}_i\bar{B}_i^T X_i = 0 \ \ (i = 1, \dots, m) \tag{2.23}$$

then

$$G_K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \in \mathbb{RH}_\infty^{m\times m} \tag{2.24}$$

is an inner function.

In order to prove this theorem, the following lemma is required.

Lemma 2.3.1 *[27] Let*

$$U_j(s) \quad = \quad \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] \in \mathbb{RH}_\infty^{p\times q} \tag{2.25}$$

where $p \geq q$, assume that rank $D_u = q$, *(C_u, A_u) is detectable. If there exist $X = 0$ satisfying*

$$XA_u + A_u^T X + C_u^T C_u \quad = \quad 0 \tag{2.26}$$
$$D_u C_u + B_u^T X \quad = \quad 0 \tag{2.27}$$
$$D_u^T D_u \quad = \quad I, \tag{2.28}$$

then A_u is stable and $U_j(s)$ is an inner function.

Using Lemma 2.3.1, we show the proof of Theorem 2.3.1.

proof 2.3.1 *We show that $G_{Ki}(s)(i = 1, \ldots, m)$ in (2.47) satisfies conditions in Lemma 2.3.1. From (2.13), (C_{Ki}, A_i) is detectable and* rank $D_i = 1$. *From Lemma 2.3.1, if $G_{Ki}(s)(i = 1, \ldots, m)$ satisfies (2.26), (2.27) and (2.28), then $G_{Ki}(s)(i = 1, \ldots, m)$ is an inner function. If $G_{Ki}(s)(i = 1, \ldots, m)$ is an inner function, $G_K(s)$ in (2.16) is a diagonal inner function.*

The rest is to prove that $G_{Ki}(s)(i = 1, \ldots, m)$ in (2.13) satisfies(2.26), (2.27) and (2.28). By substitution of $G_{Ki}(s)$ in (2.13) to (2.26), we have (2.23). From (2.22), $G_i(s)$ holds (2.27). In addition $D_{Ki} = 1(i = 1, \ldots, m)$, (2.28)is satisfied. Hence, $G_{Ki}(s)$ satisfies all conditions in (2.26), (2.27) and (2.28).

We have thus proved Theorem 2.3.1.

This method can apply to the bi-proper system $G(s)$. However, this method cannot apply to strictly proper systems. Next, we present a diagonal-inner outer factorization for strictly proper systems.

2.4 Problem formulation for strictly proper systems

This section provides an innovative problem formulation tailored for strictly proper systems. We identify and address specific challenges, incorporating a refined set of variables and methodologies that yield clarity with these systems.

Consider a linear time-invariant systems of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. , \tag{2.29}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^m$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. It is assumed that all eigenvalues of A are in the open left half plane, (A, B) is stabilizable, (C, A) is detectable, the system in (2.29) satisfies

$$\text{rank}\, G(s) = m. \tag{2.30}$$

In addition, it is assumed that the system in (2.1) has no zero on the imaginary axis, that is,

$$\text{rank} \left[\begin{array}{cc} A - sI & B \\ C & 0 \end{array} \right] = m \ \ \forall s = j\omega(-\infty < \omega < \infty). \tag{2.31}$$

It is assumed that unstable zeros of the system in (2.29) are row unstable zeros. That is, all unstable zeros z of $G(s)$ are located as

$$\text{rank} \left[\begin{array}{cc} A - zI & B \\ C_i & 0 \end{array} \right] < m(i = 1, \ldots, m), \tag{2.32}$$

where

$$C = \left[\begin{array}{c} C_1 \\ \vdots \\ C_m \end{array} \right]. \tag{2.33}$$

The transfer function from $u(s)$ to $y(s)$ in (2.29) is denoted by

$$\begin{aligned} G(s) \quad &= \quad C(sI - A)^{-1}B \in \mathbb{R}^{m\times m}(s) \\ &= \quad \left[\begin{array}{c|c} A & B \\ \hline C & 0_{m\times m} \end{array}\right] \end{aligned} \tag{2.34}$$

The system in (2.29) is assumed to satisfy

$$\text{rank}\ \Phi = m, \tag{2.35}$$

where

$$\Phi = \begin{bmatrix} B_1^T (A^T)^{\alpha_1 - 1} C^T \\ \vdots \\ B_m^T (A^T)^{\alpha_m - 1} C^T \end{bmatrix}, \tag{2.36}$$

$$B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}\ (B_i \in \mathbb{R}^n, i = 1, \cdots, m) \tag{2.37}$$

and

$$\alpha_i = \min \left(j | B_i^T \left(A^T\right)^{j-1} C^T \neq 0; j = 1, \cdots, n \right) (i = 1, \cdots, m). \tag{2.38}$$

An additional problem in this thesis is to clarify diagonal-inner outer factorization of the form in (2.8), where $G_i(s) \in \mathbb{R}^{m \times m}(s)$ is a diagonal-inner function and $G_o(s) \in \mathbb{R}^{m \times m}(s)$ is an outer function.

2.5 Diagonal-inner outer factorization for strictly proper systems

In this section, we unveil a specialized diagonal-inner outer factorization, optimized for strictly proper systems. The innovation lies in its significance in methods and offering tangible improvements in the system $G(s)$ in (2.29).

According to [28], if (2.35) is satisfied, there exists $\bar{G}(s)$ satisfying

$$\bar{G}(s) G(s) = Q(s) \tag{2.39}$$

and

$$\text{rank}\ \bar{D} = m, \tag{2.40}$$

where

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right], \tag{2.41}$$

$\bar{A} \in \mathbb{R}^{n \times n}$, $\bar{B} \in \mathbb{R}^{n \times m}$, $\bar{C} \in \mathbb{R}^{m \times n}$ and $\bar{D} \in \mathbb{R}^{m \times m}$ and

$$Q(s) = \text{diag} \left\{ \begin{array}{ccc} \frac{1}{(1 + sT_1)^{\alpha_1}} & \cdots & \frac{1}{(1 + sT_m)^{\alpha_m}} \end{array} \right\}, \tag{2.42}$$

and

$$T_i > 0 \in \mathbb{R}\ (i = 0, \cdots, m). \tag{2.43}$$

$\bar{G}(s)$ satisfying (2.39) is factorized by

$$\bar{G}(s) = \begin{bmatrix} \bar{G}_1(s) & \ldots & \bar{G}_m(s) \end{bmatrix}, \tag{2.44}$$

where $\bar{G}_i(s) \in \mathbb{RH}_\infty^m (i = 1, \ldots, m)$. Minimum realization of $\bar{G}_i(s) (i = 1, \ldots, m)$ in (2.44) is denoted by

$$\bar{G}_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array} \right] (i = 1, \ldots, m), \tag{2.45}$$

where $\bar{A}_i \in \mathbb{R}^{n_i \times n_i} (i = 1, \ldots, m)$, $\bar{B}_i \in \mathbb{R}^{n_i \times 1} (i = 1, \ldots, m)$, $\bar{C}_i \in \mathbb{R}^{m_i \times n_i} (i = 1, \ldots, m)$ and $\bar{D}_i \in \mathbb{R}^{m_i \times 1} (i = 1, \ldots, m)$. Using $K_i \in \mathbb{R}^{1 \times n_i} (i = 1, \ldots, m)$ to make $\bar{A}_i - \bar{B}_i K_i (i = 1, \ldots, m)$ have no eigenvalue in the closed right half plane, we have

$$\begin{aligned} \left[\begin{array}{c|c} \bar{A}_i - \bar{B}_i K_i & \bar{B}_i \\ \hline \bar{C}_i - \bar{D}_i K_i & \bar{D}_i \end{array} \right] &= \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array} \right] \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array} \right]^{-1} \\ &= \bar{G}_i(s) G_{Ki}(s), \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} G_{Ki}(s) &= \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|c} \bar{A}_i - \bar{B}_i K_i & \bar{B}_i \\ \hline -K_i & 1 \end{array}\right]. \end{aligned} \tag{2.47}$$

This yields

$$\hat{G}(s) = \bar{G}(s) G_K(s), \tag{2.48}$$

where

$$\begin{aligned} &\hat{G}(s) \\ &= \left[\begin{array}{ccc} \bar{C}_1 \left(sI - \bar{A}_1 + \bar{B}_1 K_i\right)^{-1} \bar{B}_1 + \bar{D}_1 & \dots & \bar{C}_m \left(sI - \bar{A}_m + \bar{B}_m K_m\right)^{-1} \bar{B}_m + \bar{D}_m \end{array} \right], \end{aligned} \tag{2.49}$$

$$\begin{aligned} G_K(s) &= \mathrm{diag} \left[\begin{array}{ccc} \left\{1 + K_1 \left(sI - \bar{A}_1\right)^{-1} \bar{B}_1\right\}^{-1} & \dots & \left\{1 + K_m \left(sI - \bar{A}_m\right)^{-1} \bar{B}_m\right\}^{-1} \end{array} \right] \\ &= \left[\begin{array}{ccc|ccc} \bar{A}_1 - \bar{B}_1 K_1 & & 0 & \bar{B}_1 & & \\ & \ddots & & & \ddots & \\ 0 & & \bar{A}_m - \bar{B}_m K_m & 0 & & \bar{B}_m \\ \hline -K_1 & & 0 & 1 & & \\ & \ddots & & & \ddots & \\ 0 & & -K_m & 0 & & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \end{aligned} \tag{2.50}$$

and

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array} \right]. \tag{2.51}$$

From (2.49), we have

$$G(s) = G_K(s) \hat{G}^{-1}(s) Q(s). \tag{2.52}$$

From the discussion in section 2.3 and Theorem 2.3.1, if $K_i (i = 1, \dots, m)$ is settled by (2.22), $G_K(s)$ in (2.50) is a diagonal-inner function. From (2.15), $\hat{G}(s)$ is stable and of minimum phase. Therefore $\hat{G}^{-1}(s)Q(s)$ is stable and of minimum phase. Since $G(s)$ is strictly proper and $G_K(s)$ is bi-proper, $\hat{G}^{-1}(s)Q(s)$ is strictly proper. Thus $\hat{G}^{-1}(s)Q(s)$ is an outer function. In this way, $G(s)$ is factorized as

$$G(s) = G_i(s) G_o(s), \tag{2.53}$$

where $G_i(s)$ is an inner function and given by

$$G_i(s) = G_K(s) \tag{2.54}$$

and $G_o(s)$ is an outer function and given by

$$G_o(s) = G_i^{-1} G(s) = \hat{G}^{-1}(s) Q(s). \tag{2.55}$$

In addition, from the assumption that all unstable zeros of $G(s)$ satisfy (2.32), and (2.55), $G_o(s)$ is obviously stable and of minimum-phase.

2.6 State space design method of inter-outer factorization for strictly proper systems

In this section, we explain state space design method for the diagonal-inner outer factorization described in the previous section. Illustrated through concrete equations to use for an example in the next section. This method showcases its calculation efficiency in deploying for system.

According to [28] and easy calculations, the state space representation of $\bar{G}(s)$ satisfying (2.39) and (2.40) is written by

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{c|c} A - \Psi\hat{\Phi}^T C & \Psi\hat{\Phi}^T \\ \hline P\hat{\Phi}^T C & P\hat{\Phi}^T \end{array}\right], \tag{2.56}$$

where

$$\beta_{ij} = \begin{Bmatrix} \alpha_i \\ j \end{Bmatrix} (T_i)^{-j} \quad (i = 1, \ldots, m : j = 1, \ldots, \alpha_i), \tag{2.57}$$

$$\Phi\hat{\Phi} = I_m, \tag{2.58}$$

$$P = \mathrm{diag}\left\{ \begin{array}{ccc} \beta_{1\alpha_1} & \ldots & \beta_{m\alpha_m} \end{array} \right\}, \tag{2.59}$$

and

$$\Psi_i = \left[\begin{array}{ccc} A^{\alpha_1}B_1 + \ldots + \beta_{1\alpha_1}B_1 & \ldots & A^{\alpha_m}B_m + \ldots + \beta_{m\alpha_m}B_m \end{array} \right] (j = 1, \ldots, m). \tag{2.60}$$

From [28], we find that $\bar{G}(s)$ in (2.56). In addition, from the assumption in (2.35) and (2.58), $\bar{D}$ in (2.56) satisfies

$$\mathrm{rank}\ \bar{D} = \mathrm{rank}\ P\hat{\Phi}^T \tag{2.61}$$

$$= m. \tag{2.62}$$

Thus, it is confirmed that $\bar{G}(s)$ in (2.56) satisfies (2.39) and (2.40).

$K_i(i = 1, \ldots, m)$ in (2.47) is obtained using Theorem 2.3.1.

2.7 Numerical example

In this section, we show numerical examples to illustrate the effectiveness of the proposed method.

2.7.1 Numerical example of a bi-proper system

Let us consider the problem problem of the diagonal-inner outer factorization for the system in

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 3 & -8 & -5 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \end{array} \right. \tag{2.63}$$

using the method in section 2.3. The transfer function $G(s)$ of the system in (2.63) is given by

$$G(s) = \begin{bmatrix} \dfrac{s^2 - 1}{s^2 + 5s + 6} & \dfrac{s - 1}{s^2 + 9s + 20} \\ \dfrac{1}{s + 5} & \dfrac{s + 4}{s + 5} \end{bmatrix}. \tag{2.64}$$

$\bar{G}(s)$ satisfying (2.9) is written by

$$\bar{G}(s) = \left[\begin{array}{ccccc|cc} -5 & 8 & 5 & 0 & -6 & -1 & 0 \\ -3 & 5 & 5 & 0 & -6 & -1 & 0 \\ 0 & 0 & -4 & -1 & 1 & 0 & -1 \\ -3 & 8 & 5 & -5 & -6 & -1 & 0 \\ 0 & 0 & 0 & -1 & -4 & 0 & -1 \\ \hline 3 & -8 & -5 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]. \tag{2.65}$$

$\bar{G}(s)$ is factorized as (2.10). $\bar{G}_1(s)$ and $\bar{G}_2(s)$ are given by

$$\bar{G}_1(s) = \left[\begin{array}{ccccc|c} -5 & 8 & 5 & 0 & -6 & -1 \\ -3 & 5 & 5 & 0 & -6 & -1 \\ 0 & 0 & -4 & -1 & 1 & 0 \\ -3 & 8 & 5 & -5 & -6 & -1 \\ 0 & 0 & 0 & -1 & -4 & 0 \\ \hline 3 & -8 & -5 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

and

$$\bar{G}_2(s) = \left[\begin{array}{cccc|c} -0.94260 & 0 & 0 & 0 & -0.50650 \\ 0 & -5.72300 & 0 & 0 & 0.81630 \\ 0 & 0 & -3.66700 & 0.51850 & -0.87820 \\ 0 & 0 & -0.51850 & -3.66700 & -0.09770 \\ \hline 0.47390 & -0.41790 & 0.40220 & 0.67190 & 0 \\ -0.15500 & -0.24250 & -1.27000 & -0.03965 & 1 \end{array}\right]$$

From (2.22), $K_i (i = 1, 2)$ are calculated by

$$K_1 = \begin{bmatrix} 2 & -4 & -2 & 0 & 2 \end{bmatrix} \tag{2.66}$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.67}$$

Using above mentioned parameters and (2.13), Thus $G_K(s)$ is given by

$$G_K(s) = \left[\begin{array}{cc|cc} 1 & 0 & 1.51200 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1.32300 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{2.68}$$

From (2.15) $\hat{G}_j (j = 1, 2)$ is given by

$$\hat{G}_1(s) = \left[\begin{array}{ccccc|c} -5.72300 & 0 & 0 & 0 & 0 & -0.29270 \\ 0 & -3.6670 & 0.51850 & 0 & 0 & -0.01287 \\ 0 & -0.51850 & -3.66700 & 0 & 0 & -0.06238 \\ 0 & 0 & 0 & -0.94260 & 0 & 24.97000 \\ 0 & 0 & 0 & 0 & -1 & 24.40000 \\ \hline 0.73250 & -0.34930 & -1.47700 & 1.56700 & -1.47500 & 0 \\ 0.42510 & 2.32400 & 0.81800 & -0.51250 & 0.49170 & 1 \end{array}\right]$$

$$\hat{G}_2(s) = \left[\begin{array}{cccc|c} -0.94260 & 0 & 0 & 0 & -0.50650 \\ 0 & -5.72300 & 0 & 0 & 0.81630 \\ 0 & 0 & -3.66700 & 0.51850 & -0.87820 \\ 0 & 0 & -0.51850 & -3.66700 & -0.09770 \\ \hline 0.47390 & -0.41790 & 0.40220 & 0.67190 & 0 \\ -0.15500 & -0.24250 & -1.27000 & -0.03965 & 1 \end{array}\right]$$

Thus the diagonal-inner outer factorization for $G(s)$ in (2.63) is obtained as

$$G_i(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} G_o(s) &= \left[\begin{array}{ccccc|cc} -2 & 0 & 0 & 0 & 0 & -0.65690 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1.37000 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0.74450 \\ 0 & 0 & 0 & -5 & 0 & 2.12100 & -3.23800 \\ 0 & 0 & 0 & 0 & -5 & -1.70400 & 4.13110 \\ \hline -1.52200 & -2.91900 & -4.02900 & 2.10200 & 2.61600 & 1 & 0 \\ 0 & 0 & 0 & 0.74810 & 0.34430 & 0 & 1 \end{array}\right] \\ &= \begin{bmatrix} \frac{s^4+12s^3+46s^2+60s+25}{s^4+15s^3+81s^2+185s+150} & \frac{s^2+6s+5}{s^3+14s^2+65s+100} \\ \frac{s+5}{s^2+10s+25} & \frac{s^2+9s+20}{s^2+10s+25} \end{bmatrix}. \end{aligned} \tag{2.69}$$

From (2.69), $G_i(s)$ is obviously a diagonal-inner function. Since $G_o(s)$ in (2.69) is stable and zeros of $G_o(s)$ in (2.69) are located in -5.7230, -0.9426, -1, $-3.6672 + 0.5185j$ and $-3.6672 - 0.5185i$, $G_o(s)$ in (2.69) is an outer function.

In this way, it is easily to obtain the diagonal-inner outer factorization using the method in section 2.3.

2.7.2 Numerical example of a strictly proper system

Consider the problem of obtaining the diagonal-inner outer factorization for the system in

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -4 & 5 & 0 & 0 & -7 & 0 & 8 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t) \end{cases}. \quad (2.70)$$

The transfer function $G(s)$ of the system in (2.70) is given by

$$G(s) = \begin{bmatrix} \dfrac{s-3}{s^2+3s+2} & \dfrac{s-3}{s^2+9s+20} \\ \dfrac{1}{s^2+11s+30} & \dfrac{s+1}{s^2+5s+6} \end{bmatrix}. \quad (2.71)$$

From (2.38), since

$$B_1^T C^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (2.72)$$

and

$$B_2^T C^T = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad (2.73)$$

we have

$$\alpha_1 = 1 \quad (2.74)$$

and

$$\alpha_2 = 1. \quad (2.75)$$

From (2.36) and (2.58), Φ and $\hat{Phi}$ are given by

$$\Phi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$\hat{\Phi} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$T_i (i = 1, 2)$ are assigned as

$$T_1 = 0.01 \quad (2.76)$$

and

$$T_2 = 0.02. \quad (2.77)$$

Equation (2.57), (2.59) and (2.60) yield

$$\beta_{11} = 100, \quad (2.78)$$

$$\beta_{21} = 50, \quad (2.79)$$

$$P = \begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix}, \quad (2.80)$$

and

$$\Psi = \begin{bmatrix} 99 & 0 \\ 98 & 0 \\ 0 & 48 \\ 0 & 47 \\ 0 & 46 \\ 95 & 0 \\ 0 & 45 \\ 94 & 0 \end{bmatrix}. \tag{2.81}$$

$\bar{G}(s)$ satisfying (2.56) is written by

$$\bar{G}(s) = \left[\begin{array}{cccccccc|cc} 395 & -495 & -99 & 198 & 693 & 99 & -792 & -99 & -99 & 99 \\ 392 & -492 & -98 & 196 & 686 & 98 & -784 & -98 & -98 & 98 \\ 0 & 0 & 46 & -96 & 0 & -48 & 0 & 48 & 0 & -48 \\ 0 & 0 & 47 & -97 & 0 & -47 & 0 & 47 & 0 & -47 \\ 0 & 0 & 46 & -92 & -4 & -46 & 0 & 46 & 0 & -46 \\ 380 & -475 & -95 & 190 & 665 & 90 & -760 & -95 & -95 & 95 \\ 0 & 0 & 45 & -90 & 0 & -45 & -5 & 45 & 0 & -45 \\ 376 & -470 & -94 & 188 & 658 & 94 & -752 & -100 & -94 & 94 \\ \hline -400 & 500 & 100 & -200 & -700 & -100 & 800 & 100 & 100 & -100 \\ 0 & 0 & -50 & 100 & 0 & 50 & 0 & -50 & 0 & 50 \end{array}\right]. \tag{2.82}$$

$\bar{G}(s)$ is factorized as (2.44), where

$$\bar{G}_1(s) = \left[\begin{array}{cccccccc|c} 395 & -495 & -99 & 198 & 693 & 99 & -792 & -99 & -99 \\ 392 & -492 & -98 & 196 & 686 & 98 & -784 & -98 & -98 \\ 0 & 0 & 46 & -96 & 0 & -48 & 0 & 48 & 0 \\ 0 & 0 & 47 & -97 & 0 & -47 & 0 & 47 & 0 \\ 0 & 0 & 46 & -92 & -4 & -46 & 0 & 46 & 0 \\ 380 & -475 & -95 & 190 & 665 & 90 & -760 & -95 & -95 \\ 0 & 0 & 45 & -90 & 0 & -45 & -5 & 45 & 0 \\ 376 & -470 & -94 & 188 & 658 & 94 & -752 & -100 & -94 \\ \hline -400 & 500 & 100 & -200 & -700 & -100 & 800 & 100 & 100 \\ 0 & 0 & -50 & 100 & 0 & 50 & 0 & -50 & 0 \end{array}\right]$$

and

$$\bar{G}_2(s) = \left[\begin{array}{ccccccc|c} -50 & 0 & 0 & 0 & 0 & 0 & 0 & 11.2100 \\ 0 & -100 & 0 & 0 & 0 & 0 & 0 & -132.7000 \\ 0 & 0 & -5.8040 & 2.9330 & 0 & 0 & 0 & 3.4240 \\ 0 & 0 & -2.9330 & -5.8040 & 0 & 0 & 0 & 4.8860 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0.6454 \\ 0 & 0 & 0 & 0 & 0 & -3.6960 & 0.4947 & 0.5945 \\ 0 & 0 & 0 & 0 & 0 & -0.4947 & -3.6960 & -0.6110 \\ \hline 0 & -76.1300 & -3.0280 & 2.4780 & 0 & -1.6620 & 0.01126 & -100 \\ -200 & 0 & -0.4264 & -1.5550 & 3.1890 & -0.07548 & 0.2792 & 50 \end{array}\right]. \tag{2.83}$$

From (2.22), $K_i (i = 1, 2)$ are calculated by

$$K_1 = \begin{bmatrix} -6 & 6 & 0 & 0 & -6 & 0 & 6 & 0 \end{bmatrix} \tag{2.84}$$

and

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.85}$$

Using above mentioned parameters and (2.50), Thus G_K is given by

$$G_K(s) = \left[\begin{array}{cc|cc} -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{2.86}$$

Thus the diagonal-inner outer factorization for $G(s)$ in (2.70) is obtained as

$$G_i(s) = \begin{bmatrix} \frac{s-3}{s+3} & 0 \\ 0 & 1 \end{bmatrix}$$

From (2.42) Q is given by

$$Q(s) = \left[\begin{array}{cc|cc} -100 & 0 & 8 & 0 \\ 0 & 50 & 0 & 8 \\ \hline 12.5 & 0 & 0 & 0 \\ 0 & 6.25 & 0 & 0 \end{array}\right]. \tag{2.87}$$

From (2.49), the inverse of $\hat{G}(s)$ is given by

$$\hat{G}^{-1}(s) = \left[\begin{array}{cccccccc|cc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.1120 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.8790 \\
0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0290 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2.1200 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1.0960 & -0.4194 \\
0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 2.0770 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 1.0060 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 2.3940 & 0 \\
\hline
-1.7800 & 0 & -0.8943 & 0.1769 & 0.8943 & 0 & 1.7890 & 0 & 0.01 & 0.02 \\
0 & 0.4847 & 0 & -0.4527 & 0 & -0.4526 & 0 & 0.3969 & 0 & 0.02
\end{array}\right].$$

From (2.55), $G_o(s)$ is obtained as

$$\begin{aligned} G_o(s) &= \left[\begin{array}{cccccccc|cc}
-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.554 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.978 \\
0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & -1.069 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0.5571 & 0 \\
0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0.0003145 & 0.554 \\
0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & -1.207 & 0.0006272 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -0.5538 & -0.2159 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -0.5375 \\
\hline
1.805 & 0 & 0 & 3.59 & 3.61 & 0.0009405 & 1.806 & -0.7255 & 0 & 0 \\
0 & 2.045 & 0.9351 & 0 & 0.0009378 & -0.8282 & -0.0001021 & 1.861 & 0 & 0
\end{array}\right] \\
&= \begin{bmatrix} \dfrac{s^3+13s^2+55s+75}{s^4+13s^3+57s^2+95s+50} & \dfrac{s^4+12s^3+51s^2+92s+60}{s^5+18s^4+125s^3+416s^2+660s+400} \\ \dfrac{1.001s^2+7.005s+10.01}{s^4+18s^3+117s^2+320s+300} & \dfrac{s^4+13s^3+57s^2+95s+50}{s^5+17s^4+111s^3+347s^2+520s+300} \end{bmatrix} \end{aligned} \tag{2.88}$$

From (2.87), $G_i(s)$ is obviously a diagonal-inner function. Since $G_o(s)$ in (2.88) is stable and zeros of $G_o(s)$ in (2.88) are located in $-5.8043+2.9331j$, $-5.8043-2.9331j$, -1, $-3.6957+0.4947j$, $-3.6957-0.4947j$, -3, $G_o(s)$ in (2.87) is an outer function.

In this way, The diagonal-inner outer factorization can be readily attained by utilizing the methodology delineated in section 2.5.

2.8 Conclusions.

Table 2.1: Comparison of inner outer factorization method

Referent	Strictly proper	Bi-proper	Diagonal-inner function
W. Kase et al. [22]	✓	✗	✗
G. Gu [20]	✓	✓	✗
A. Varga [2]	✓	✓	✗
This thesis	✓	✓	✓

In this thesis, we propose a method of diagonal-inner outer factorization for bi-proper systems and strictly proper systems. Comparisons between our method and past research [2, 20, 22] are summarized in Table 2.1. it is evident that a number of studies have engaged in inner outer factorization, predominantly focusing on the factorization of strictly proper systems. There also exists a subset of research that delves into the factorization of bi-proper systems. However, a discernible gap in the literature is the absence of work incorporating the factorization method yielding a diagonal-inner function. Hence, the exploration and articulation of diagonal-inner function constitute a significant contribution and distinction of the present research.

The application adoption of this method is expected to be a pivotal tool in augmenting the performance of control systems within industrial manufacturing contexts. By refining the control algorithms, there is a substantial potential for industries to realize enhanced operational efficiency. The integration with artificial intelligence and machine learning is anticipated, facilitating advanced system control and adaptability. Furthermore, this integration is foreseen to extend to emerging Industrial Internet of Things (IIoT) applications, engendering real-time control and optimized performance.

The thesis concludes by re-emphasizing has proposed method of diagonal-inner outer factorization for bi-proper systems and for strictly proper systems. Basic idea for diagonal-inner outer factorization is explained based on transfer function. In addition, we present a design method using state-space representation. The proposed method provides a diagonal inner function. The numerical example is also depicted in order to illustrate the effectiveness of the proposed method. Application of this thesis will be presented in another article.

Chapter 3

Conclusions

This thesis has undertaken a detailed exploration into the realm of diagonal-inner outer factorization, introducing a novel approach where the inner function is specifically a diagonal function. Through rigorous research and analysis, the study has traversed the landscape of matrix factorization, delving into its historical evolution, examining the state-of-the-art methodologies, and critically evaluating the existing techniques. The primary contribution of this research lies in the development and proposal of a new factorization method designed to overcome the limitations in efficiency and adaptability found in conventional methods. A significant finding of this study is the demonstrable increase in computational efficiency achieved by implementing a diagonal function as the inner component of factorization. Theoretically, this research challenges the traditional paradigms of factorization, advocating for a more nuanced and structure-conscious approach, thereby enriching the existing body of knowledge in computational mathematics. Beyond the academic sphere, the practical implications of this research are manifold. The proposed method holds promise for enhanced applications in various fields such as numerical analysis, engineering, data processing, and scientific computing. By optimizing the factorization process, the method paves the way for more efficient computational models and algorithms, potentially impacting how complex mathematical problems are approached and solved in various industries.

As this field continues to evolve, several key areas for future research have emerged from this thesis. For example, Topic Comprehensive Comparative Analysis: Conducting in-depth comparative studies with existing factorization techniques would offer a robust evaluation of the proposed method. This could involve quantitative performance metrics, computational resource usage assessments, and accuracy benchmarks, providing a well-rounded perspective of the method's strengths and areas for improvement. Topic Synergy with Advanced Computational Fields: Exploring the integration of the proposed factorization method with cutting-edge fields such as machine learning, artificial intelligence, and big data analytics could yield transformative results. This integration could lead to innovative approaches in data interpretation, algorithm development, and complex problem-solving. Topic Advancements in Computational Efficiency: Investigating further into optimizing the computational aspects of the proposed method remains a promising avenue. This could encompass the development of parallel computing strategies, algorithmic enhancements, and the utilization of advanced computing platforms to elevate the method ’ s performance and scalability.

In closing, this thesis stands as a pivotal contribution to the field of diagonal-inner outer factorization, challenging existing norms and introducing an innovative approach with far-reaching theoretical and practical implications. As the demands in computational mathematics continue to grow and diversify, the insights and methodologies presented in this research offer valuable assets for advancing efficient and sophisticated mathematical solutions.

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Publication lists

Chapter 2

- ○ S.Fong-In, P.Ketthong, N. T. Mai, K.Hashikura, M. A. S. Kamal,I.Murakami, W.San-um and K.Yamada, A NOVEL METHOD OF DIAGONAL-INNER OUTER FACTORIZATION, International Journal of Innovative Computing, Information and Control (IJICIC), 2023.
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