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A generalization of Morita duality by localizations

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Abstract

Let R and S be rings with identity, and *Mod-R* and *S-Mod* the category of unital right R- and left S-modules, respectively. Also let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of *Mod-R* and *S-Mod* such that $\mathcal{A} \ni R$ and $\mathcal{B} \ni S$ and both are closed under finite direct sums, submodules and epimomorphic images. We will find conditions in order that there exists a duality between Giraud subcategories of $\mathcal{A}$ and $\mathcal{B}$. As an application of this we will obtain a general result of [8] about Morita duality between Grothendieck categories.

Introduction

Morita equivalence and Morita duality have been generalized by many authors. Kato's idea [12] is important as a generalization of Morita equivalence, because his idea seems to be useful to get a generalization of Morita duality. Kato's result has two important things. The one thing is to give concrete localization and colocalization functors. The another thing, which is a direct result of the preceding fact, is to give a category equivalence as a generalization of Morita equivalence. If we dualize his idea then we get a duality (contravariant equivalence) between localizations of abelian categories. The author tried to get a nontrivial example in [22].

Morita duality is a duality between full subcategories of module categories. It is natural to define a Morita duality between abelian categories as a duality between full subcategories of them. Colby-Fuller [4] and Anh-Wiegandt [2] defined Morita dualities from this point. Quite recently Gómez and Guil [8] defined a QF-3″ ring and obtained a Morita duality between Grothendieck categories in the sense of Anh-Wiegandt. Almost at the same time the author defined QF-3″ object of an abelian category. But this definition already appeared in [21] as the different name. Later we will show that a ring R is left QF-3″ iff R is a QF-3″ object in $\overline{R\text{-}mod}$, the full subcategory of *R-Mod* consisting of all submodules of finitely generated left R-modules. We will have the Morita duality in the sense of Anh-Wiegandt obtained by Gómez and Guil as a special case of our result.

In [29] Tachikawa and the author formally established a category equivalence between Giraud subcategories and co-Giraud subcategories of abelian categories (these are category equivalent to quotient categories factored by strongly hereditary torsion classes and strongly cohereditary torsion free classes, respectively). In [21] the author defined a CQF-3 object in an abelian category and applied this notion to a cocomplete abelian category. QF-3″ object is a dual version of this notion. Cocompleteness was necessary to define a left adjoint of a hom-functor. In the original Morita duality defined by a bimodule ${}_SU_R$, the category of U-reflexive modules does not contain infinite direct sums. Thus one of the main difficulties to generalize Motita duality is to get functors between two categories.

A full subcategory of an abelian category is called strongly exact if it is closed under subobjects, quotient objects and finite coproducts. Then a strongly exact subcategory of an abelian category is clearly abelian. Let $\mathcal{A}$ and $\mathcal{B}$ be strongly exact subcategories of *Mod-R* and *S-Mod*, respectively, such that $\mathcal{A} \ni R$ and $\mathcal{B} \ni S$. In section 1 we will find a necessary and sufficient condition in order to exist a duality between given Giraud subcategories of $\mathcal{A}$ and $\mathcal{B}$. It will be obtained that such a duality is given by a bimodule ${}_SU_R$. In this case it is important that S is not necessarily an endomorphism ring of U_R. Also the notion of QF-3″ is important. Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$. Then we call A QF-3″ if for any monomorphism $f: X' \to X$, $Hom(f, A)=0$ implies $Hom_{\mathcal{A}}(X', A)=0$. We will see that if a bimodule ${}_SU_R$ defines a duality as the above then U_R and ${}_SU$ have to be QF-3″ in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Section 2 is the main part of this paper. We call a duality of the type given in section 1 a localized Morita duality (the exact definition is given in that section). If there exists a localized Morita duality then the duality itself is a Morita duality in the sense of Anh-Wiegandt between Grothendieck categories. But a localized Morita duality contains much more : it contains localizations of strongly exact subcategories of module categories and a duality between localized categories (Giraud subcategories) of those categories. We will find a necessary and sufficient condition on a bimodule ${}_SU_R$ in order that it defines a localized Morita duality. Theorems 2.3 and 2.8 are the main results of this paper. We define a QF-3″ module as a generalization of a one sided QF-3″ ring. Actually a QF-3″ *module is equivalent to a* QF-3″ *object in some strongly exact subcategory of Mod-R.* Anyway this notion plays an important role especially in Theorem 2.8.

If R is a QF-3 ring with minimal faithful modules eR and Rf. Then as was proved in

[28] ${}_{eRe}R_{fRf}$ defines a Morita duality. In section 3 will consider a generalization of this fact as an application of our result.

1. Localized Morita duality

We assume that the readers are familiar with torsion theories and localizations in abelian categories. For the terminologies in this paper please refer to [21], [26] and [29]. A strongly exact subcategory of an abelian category is a full subcategory closed under subobjects, quotient objects and finite coproducts. The following result seems to be well known.

LEMMA 1.1. *Let $\mathcal{A}$ be a strongly exact subcategory of Mod-R and $(\mathcal{T}, \mathcal{F})$ a torsion theory in $\mathcal{A}$. Suppose $\mathcal{A} \ni R$ and R has its localization $\phi : R \to R'$. Then*

(i) R' is given a ring structure such that ϕ is a ring homomorphism.

(ii) If $X \in \mathcal{A}$ is torsion free divisible then X can be seen as an R'-module.

(iii) If X and Y are torsion free divisible then there is a canonical isomorphism $Hom_R(X, Y) \simeq Hom_{R'}(X, Y)$.

We already defined a QF-3″ object of an abelian category. The next lemma is crucial in this paper. The author believes that the idea is good. When the author contributed this paper to some journal the referee ordered to omit the proof since there were no new result. The referee's opinion is absolute. So the author had to cut the proof. It was true that the author had bad feeling to the referee. And he wrote in the paper "We omit the proof of the following result bye the referee's suggestion." ... The paper was rejected. The proof is written in the paper entitled Morita duality for Grothendieck categories and its appliction, to appear in J. Algebra.

LEMMA 1.2. *Let $\mathcal{A}$ be a strongly exact subcategory of Mod-R and $U \in \mathcal{A}$. Let $t(X) = \cap \{ Ker f \mid f \in Hom_R(X, U) \}$ for $X \in \mathcal{A}$. Then the following assertions are equivalent.*

(i) U is QF-3″ in $\mathcal{A}$.

(ii) t is a left exact radical.

(iii) For any monomorphism $f : X' \to X$ in $\mathcal{A}$, if $g : X' \to U$ is nonzero, then there exists $s \in S := End_R(U)$ and $h : X \to U$ such that $h \cdot f = s \cdot g \neq 0$.

(iv) If $U \trianglelefteq Y$ is an essential extension of U_R and $Y \in \mathcal{A}$ then $t(Y) = 0$.

(v) For any $X \in \mathcal{A}$, $t(X) = \cap \{ Ker f \mid f \in Hom_R(X, E(U_R)) \}$, where $E(U_R)$ denotes the

injective envelope of U_R.

The hereditary torsion theory associated to the left exact radical in the preceding lemma is said to be cogenerated by U_R.

THEOREM 1.3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be strongly exact subcategories of Mod-R and S-Mod such that* $\mathcal{A}\ni R$ *and* $\mathcal{B}\ni S$, *respectively. Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be Giraud subcategories of* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively. Then there exists a duality between* $\mathcal{C}$ *and* $\mathcal{C}'$ *if and only if there exists a bimodule* ${}_SU_R$ *which satisfies the following conditions.*

(i) *For any* $X\in\mathcal{A}$ *and* $Y\in\mathcal{B}$, $Hom_R(X, U)\in\mathcal{B}$ *and* $Hom_S(Y, U)\in\mathcal{A}$.

(ii) *For any* $X\in\mathcal{A}$ *and* $Y\in\mathcal{B}$, *the canonical homomorphisms*

$$\eta_X : X\to Hom_S(Hom_R(X, U), U) \text{ and}$$
$$\eta_Y : Y\to Hom_R(Hom_S(Y, U), U)$$

give localizations for X *and* Y *with respect to the torsion theories corresponding to* $\mathcal{C}$ *and* $\mathcal{C}'$, *respectively.*

Moreover if these two conditions are satisfied then U_R *and* ${}_SU$ *are QF-3″ in* $\mathcal{A}$ *and* $\mathcal{B}$, *and the hereditary torsion theories cogenerated by* U_R *in* $\mathcal{A}$ *and* ${}_SU$ *in* $\mathcal{B}$ *coincide with the ones corresponding to* $\mathcal{C}$ *and* $\mathcal{C}'$, *respectively. Moreover* U_R *and* ${}_SU$ *are divisible with respect to each torsion theory. Finally in this case* $Hom_R(-, U)$ *and* $Hom_S(-, U)$ *induce a duality between* $\mathcal{C}$ *and* $\mathcal{C}'$.

Proof. Suppose there are contravariant functors $F: \mathcal{C}\to\mathcal{C}'$ and $G: \mathcal{C}'\to\mathcal{C}$ which define a duality between $\mathcal{C}$ and $\mathcal{C}'$. Let $i: \mathcal{C}\to\mathcal{A}$ and $j: \mathcal{C}'\to\mathcal{B}$ be the inclusion functors and $a: \mathcal{A}\to\mathcal{C}$ and $b: \mathcal{B}\to\mathcal{C}'$ the reflectors. Then for any $X\in\mathcal{A}$ and $Y\in\mathcal{B}$ there are natural isomorphisms

$$\begin{aligned} Hom_S(Y, jFa(X)) &\simeq Hom_{\mathcal{C}'}(b(Y), Fa(X)) \\ &\simeq Hom_{\mathcal{C}}(GFa(X), Gb(Y)) \\ &\simeq Hom_{\mathcal{C}}(a(X), Gb(Y)) \\ &\simeq Hom_R(X, iGb(Y)). \end{aligned}$$

In particular $Hom_S(S, jFa(R)) \simeq Hom_R(R, iFb(S))$ holds. Both sides are S-R-bimodules. When we substitute $X=R$ and $Y=S$ we need to to show that the above isomorphisms are S- and R-homomorphisms. We only show that the first isomorphism is an S-homomorphism since other S- and R-homomorphisms can be proved similarly. For any $f: {}_SS\to{}_SjFa(R)$, we have a commutative diagram

$$\begin{array}{ccc} Hom_S(jFa(R),\ jFa(R)) & \xrightarrow{\phi} & Hom_{\mathcal{C}'}(bjFa(R),\ Fa(R)) \\ \downarrow{\scriptstyle Hom(f,\ jFa(R))} & & \downarrow{\scriptstyle Hom(b(f),\ Fa(R))} \\ Hom_S(S,\ jFa(R)) & \xrightarrow{\psi} & Hom_{\mathcal{C}'}(b(S),\ Fa(R)), \end{array}$$

where ϕ and ψ are natural isomorphisms. Thus $\psi(f)=b(f)\cdot g$ holds, where $g=\phi(1_{jFa(R)})$. For any $s\in S$ let $\lambda_s:{}_SS\to{}_SS$ be as $t\lambda_s=ts$ for $t\in S$. Then since $sf=\lambda_s f$, $\psi(sf)=b(sf)g=b(\lambda_s f)=b(\lambda_s)b(f)g=s(\psi(f))$. Therefore ψ is an S-homomorphism. Now let us put ${}_SU_R=iGb(S)$. Then we show that this U satisfies the conditions of the theorem. For any $X\in\mathcal{A}$,

$$\begin{aligned} Hom_R(X,\ U) &= Hom_R(X,\ iGb(S)) \\ &\simeq Hom_S(S,\ jFa(X)) \\ &\simeq jFa(X). \end{aligned}$$

Thus $Hom_R(-,\ U)\simeq jFa$. Similarly $Hom_S(-,\ U)\simeq iGb$ holds. Moreover

$$\begin{aligned} Hom_S(Hom_R(X,\ U),\ U) &\simeq iGbjFa(x) \\ &\simeq iGFa(X) \\ &\simeq ia(X) \end{aligned}$$

since $bj\simeq 1_{\mathcal{C}'}$ (*see* [26, p.213]). *So there exists a natural homomorphism*

$$\zeta_X : X\to Hom_S(Hom_R(X,\ U),\ U)$$

such that ζ_X is a localization for each $X\in\mathcal{A}$. However, we can not say that ζ_X coincides with η_X. We have to show that η_X also gives a localization of X. Since ${}_SU\simeq{}_SHom_R(R,\ U)$, we may assume $\zeta_R : R\to End_S(U)$. Since $\zeta : 1_{\mathcal{A}}\to Hom_S(Hom_R(-,\ U),\ U)$ is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\zeta_R} & End_S(U) \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle Hom(Hom(\phi_x,\ U),\ U)} \\ X & \xrightarrow{\zeta_X} & Hom_S(Hom_R(X,\ U),\ U), \end{array}$$

where ϕ_x is defined by $x\in X$ as $\phi_x(r)=xr$ for $r\in R$. Hence $\zeta_X(x)=\eta_X(x)\zeta_R(1)$ holds. This implies that $f\zeta_X(x)=f(x)\zeta_R(1)$ holds for all $f\in Hom_R(X,\ U)$. Now since ζ_X is an R-homomorphism,

$$f\zeta_X(xr) = f\zeta_X(x)r = f(x)\zeta_R(1_R)r$$
$$= f(xr)\zeta_R(1) = f(x)r\zeta_R(1),$$

that is, $f(x)\zeta_R(1)r = f(xr)\zeta_R(1) = f(x)r\zeta_R(1)$ holds for all $x \in X$ and $r \in R$. Thus in particular (if we take $X=R$) $\zeta_R(1)r = r\zeta_R(1)$ holds. On the other hand, $\zeta_U : U \to Hom_S(End_R(U), U)$ is an isomorphism. Thus for any $u \in U$, there exists $u' \in U$ such that $\zeta_U(u') = \eta_U(u)$, hence $u = (1_U)\eta_U(u) = (1_U)\zeta_U(u') = u'\zeta_R(1)$, which implies that $\zeta_R(1)$ is an epimorphism. Next we show that $\zeta_R(1)$ is a monomorphism. Suppose $u\zeta_R(1)=0$. If $u \neq 0$ then let $\Phi_u : End_S(U) \to U$ be as $\Phi_u(r') = ur'$. Then Φ_u is an R-homomorphism, and $\Phi_u(\zeta_R(R)) = u\zeta_R(R) = u\zeta_R(1)R = 0$. Hence Φ_u induces a nonzero homomorphism $\overline{\Phi_u} : End_S(U)/\zeta_R(R) \to U$. But $End_S(U)/\zeta_R(R)$ is torsion since ζ_R is a localization of R_R, and U_R is torsion free. This is a contradiction. Therefore $\zeta_R(1)$ is a unit in $End_S(U)$ and commuteswith each element of R. Now it is clear that η_X is a localization from the foll owing commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & Hom_S(Hom_R(X, U), U) \\ \downarrow 1_X & & \simeq \downarrow Hom(Hom(X, U), \zeta_R(1)) \\ X & \xrightarrow{\zeta_X} & Hom_S(Hom_R(X, U), U). \end{array}$$

Conversely suppose there exists a bimodule ${}_SU_R$ which satisfies the conditions (i) and (ii). To simplify notations we denote X^* instead of $Hom_R(X, U)$ or $Hom_S(X, U)$. First we show that for $X \in \mathcal{A}$, $\eta_X = 0$ iff $X^* = 0$. If $X^* = 0$ then clearly $\eta_X = 0$. Conversely suppose $\eta_X = 0$. Then since η_X is the localization of X, $\eta_X = 0$ is equivalent to $X^{**} = 0$. By the property of adjunctions, $1_X = (X^* \xrightarrow{\eta'_{X^*}} X^{***} \xrightarrow{(\eta_X)^*} X^*)$. Thus $X^* = 0$. Next we show that for any $X \in \mathcal{A}$, η'_{X^*} is an isomorphism. Since η_X is the localization of X, $(\eta_X)^*$ is a monomorphism. Hence $(\eta_X)^*$ is an isomorphism since $(\eta_X)^*$ is an epimorphism. This implies that η'_{X^*} is also an isomorphism. Similarly η_{Y^*} is an isomorphism for any $Y \in \mathcal{B}$. From these facts it is clear that $Hom_R(-, U)$ and $Hom_S(-, U)$ induce a duality between $\mathcal{C}$ and $\mathcal{C}'$.

Next we show that U_R and ${}_SU$ are QF-3″ in $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $0 \to X' \xrightarrow{f} X$ be exact in $\mathcal{A}$ with $f^* = 0$. Then since the localization functor $(\ )^{**}$ is left exact, $f^{**} = 0$ implies $X'^{**} = 0$. Thus $(X')^* = 0$ as we already have seen. Finally we show that U_R and ${}_SU$ are divisible. We note that η_{S^*} is an isomorphism since $1_{S^*} = (S^* \xrightarrow{\eta_{S^*}} S^{***} \xrightarrow{(\eta_S)^*} S^*)$ and $(\eta_S)^*$ is an isomorphism. On the other hand $S_R^* \simeq U_R$. This implies that U_R is divisible. Similarly ${}_SU$ is divisible. This completes the proof.

We call a duality of Theorem 1.3 a localized Morita duality. More precisely we say that

a bimodule ${}_SU_R$ defines a localized Morita duality if the following conditions are satisfied.
(i) There exist strongly exact subcategories $\mathcal{A}$ of *Mod-R* and $\mathcal{B}$ of *S-Mod* such that $\mathcal{A} \ni R$ and $\mathcal{B} \ni S$.
(ii) $U_R \in \mathcal{A}$ and ${}_SU \in \mathcal{B}$ and they are QF-3″ in $\mathcal{A}$ and $\mathcal{B}$, respectively.
(iii) For any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, $X^* \in \mathcal{B}$ and $Y^* \in \mathcal{A}$.
(iv) For any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, the canonical homomorphisms

$$\eta_X : X \to X^{**} \quad \textit{and} \quad \eta_Y : Y \to Y^{**}$$

are localizations with respect to the torsion theories cogenerated by U_R in $\mathcal{A}$ and ${}_SU$ in $\mathcal{B}$, respectively.

In this case we say that ${}_SU_R$ defines a localized Morita duality between $\mathcal{A}$ and $\mathcal{B}$. By the preceding theorem if the above conditions are satisfied, U_R and ${}_SU$ are divisible with respect to the torsion theories cogenerated by U_R in $\mathcal{A}$ and ${}_SU$ in $\mathcal{B}$, respectively.

2. Localized Morita duality by a bimodule

In section 1 we proved the necessary and sufficient condition in order to exist a localized Morita duality between $\mathcal{A}$ and $\mathcal{B}$. In order to exist such a duality it is necessary that U_R and ${}_SU$ are QF-3″ in $\mathcal{A}$ and $\mathcal{B}$ and the canonical homomorphisms $R \to End_S(U)$ and $S \to End_S(U)$ are the localizations of R and S with respect to the hereditary torsion theories cogenerated by U_R in $\mathcal{A}$ and ${}_SU$ in $\mathcal{B}$, respectively. Moreover U_R and ${}_SU$ are divisible with respect to each torsion theory. In fact these conditions are sufficient in order to exist a localized Morita duality. We prove this fact in Theorem 2.8. First we prove the following important fact.

THEOREM 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be strongly exact subcategories of Mod-R and S-Mod such that $\mathcal{A} \ni R$ and $\mathcal{B} \ni S$. Let ${}_SU_R$ be a bimodule which satisfies the following conditions.*
(i) For any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, $X^* := Hom_R(X, U) \in \mathcal{B}$ and $Y^* := Hom_S(Y, U) \in \mathcal{A}$.
(ii) U_R and ${}_SU$ are QF-3″ in $\mathcal{A}$ and $\mathcal{B}$ and both are divisible with respect to the torsion theories cogenerated by U_R in $\mathcal{A}$ and ${}_SU$ in $\mathcal{B}$, rspectively.
(iii) The canonical homomorphisms $R \to End_S(U)$ and $S \to End_R(U)$ are the localizations of R and S with respect to the torsion theories stated in (ii), respectively.

Let $\mathcal{A}'$ and $\mathcal{B}'$ be full subcategories of $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A}' = \{ X \in \mathcal{A} \mid \eta_X \text{ is the localization of } X \} \quad \text{and}$$

$$\boldsymbol{\mathcal{B}}' = \{ Y \in \boldsymbol{\mathcal{B}} \mid \eta'_Y \text{ is the localization of } Y \},$$

where $\eta_X : X \to X^{**}$ *and* $\eta'_Y : Y \to Y^{**}$ *are canonical homomorphisms and localizations are the ones with respect to the torsion theories stated in (ii).*

Then $\boldsymbol{\mathcal{A}}'$ *and* $\boldsymbol{\mathcal{B}}'$ *are strongly exact subcategories of Mod-R and S-Mod, respectively.*

To prove the theorem we need several lemmas. When we say something about a torsion theory we mean the one cogenerated by U_R in $\boldsymbol{\mathcal{A}}$ or the one cogenerated by ${}_SU$ in $\boldsymbol{\mathcal{B}}$.

LEMMA 2.2. *Let the situation be the same as the theorem. Then for any* $X \in \boldsymbol{\mathcal{A}}$, ${}_S(X^*)$ *is torsion free divisible.*

Proof. This is immediate from the adjoint relation $Hom_S(Y, Hom_R(X, U)) \simeq Hom_R(X, Hom_S(Y, U))$.

LEMMA 2.3. *(cf. [23, Lemma 6.4]) Let* $f : X' \to X$ *be a monomorphism in* $\boldsymbol{\mathcal{A}}$. *Then* $Hom_S(Coker\ Hom(f, U), E({}_SU)) = 0$. *Thus in particular it holds that* $(Coker\ Hom(f, U))^* = 0$.

Proof. We may assume that f is an inclusion map i. It is enough to show that if $\Phi : {}_S(X')^* \to E({}_SU)$ satisfies $(g \mid X')\Phi = 0$ for all $g \in (X)^*$ then $\Phi = 0$. Suppose $\Phi \neq 0$ then there exists $h \in (X')^*$ such that $(h)\Phi \neq 0$. Since ${}_SU \trianglelefteq E({}_SU)$, there exists $s \in S$ such that $0 \neq s((h)\Phi) = (sh)\Phi \in {}_SU$. Thus at first we may assume $0 \neq (h)\Phi \in {}_SU$. Let

$$\begin{array}{ccc} X' & \xrightarrow{\ i\ } & X \\ \downarrow{\scriptstyle h} & & \downarrow \\ U & \longrightarrow & Q \end{array}$$

be the push out of i and h. Let t be the torsion radical of $\boldsymbol{\mathcal{A}}$ associated to the torsion theory cogenerated by U_R in $\boldsymbol{\mathcal{A}}$. Then $U \cap t(Q) = 0$ since U_R is torsion free. Thus $U \to Q \to Q/t(Q)$ is a monomorphism. Now since $\cap \{ Ker\ g \mid g \in (Q/t(Q))^* \} = 0$, there exists $Q/t(Q) \to U$ such that $s'(h\Phi) \neq 0$, where we put $s' = (U \to Q \to Q/t(Q) \to U)$ (all homomorphisms appeared above). We note that $s' \in S' := End_R(U)$. *Let* $k = (X \to Q \to Q/t(Q) \to U)$ (again all homomorphisms appeared above). By Lemma 1.1 Φ is an S'-homomorphism. Hence $0 \neq s'(h\Phi) = (s'h)\Phi$. But $s'h = k \mid X'$. Hence $(s'h)\Phi = (k \mid X')\Phi = 0$ holds. This contradiction proves the lemma.

LEMMA 2.4. *Let the situation be the same as the theorem. Let* $0 \to X' \to X \to X'' \to T \to 0$ *be* *exact in $\mathcal{A}$ with T torsion. Then by applying $(\ )^*$ we obtain the exact sequence $0 \to X''^* \to X^* \to X'^* \to T' \to 0$ in $\mathcal{B}$ with T' torsion.*

Proof. The first sequence can be decomposed into two exact sequences

$$0 \to X' \to X \to I \to 0 \quad \text{and} \quad 0 \to I \to X'' \to T \to 0.$$

Then since U_R is torsion free divisible, we get $X''^* \simeq I^*$. By Lemma 3.3, $0 \to I^* \to X^* \to X'^* \to T' \to 0$ is exact with T' torsion. Then by combining with $X''^* \simeq I^*$, we get the desired exact sequence.

Corollary 2.5 (cf. [4]). *Let the situation be the same as the theorem. Then the U-double dual functors $(\ )^{**} : \mathcal{A} \to \mathcal{A}$ and $(\ )^{**} : \mathcal{B} \to \mathcal{B}$ are left exact.*

Now we are ready to prove our main theorem. It is enough to show that if $0 \to X' \to X \to X'' \to 0$ is exact in $\mathcal{A}$ and η_X is a localization then $\eta_{X'}$ and $\eta_{X''}$ are also localizations. Consider the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & K' & & K & & K'' & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X'^{**} & \longrightarrow & X^{**} & \longrightarrow & X''^{**} & \longrightarrow & T'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & L' & & L & & L'' & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

where columns and two middle rows are exact and T'' is torsion. In order to show that $\eta_{X'}$ is a localization we only need to show that L' is torsion. By snake lemma ([23, Corollary 11.9]), there exists a homomorphism $K'' \to L'$ such that the sequence $K'' \to L' \to L$ is exact. Then since the torsion class is closed under factors, submodules and group extensions in $\mathcal{A}$, we see that L' is torsion. Next we show that $\eta_{X''}$ is a localization. By applying $(\ )^*$ to the commutative diagram

$$
\begin{array}{ccccccc}
X & \longrightarrow & X'' & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & & & \\
X^{**} & \longrightarrow & X''^{**} & \longrightarrow & T'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & L'' & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

we obtain the commutative diagram

$$
\begin{array}{ccccc}
X^{*} & \longleftarrow & X''^{*} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & \\
X^{***} & \longleftarrow & X''^{***} & \longleftarrow & 0 \\
 & & \uparrow & & \\
 & & L''^{*} & & \\
 & & \uparrow & & \\
 & & 0 & &
\end{array}
$$

with exact rows and columns. Hence $L''^{*}=0$. This proves that $\eta_{X''}$ is a localization. This completes the proof.

If a module is an epimorphic image of a finite direct sum of copies of M_R then it is called finitely M_R-generated.

DEFINITION. *U_R is a QF-3″ module if it cogenerates every finitely $(U\oplus R)_R$-generated submodule of $E(U_R)$.*

For any module M_R, $\overline{cog}(M_R)$ denotes the full subcategory of *Mod-R* consisting of all submodules of finitely M_R-generated modules (or equivalently all factor modules of submodules of finite direct sums of copies of M_R).

LEMMA 2.6. *U_R is a QF-3″ module if and only if U is a QF-3″ object in $\overline{cog}((U\oplus R)_R)$.*

Proof. Suppose U_R is a QF-3″ module. Let $U\trianglelefteq X$ be an essential extension with $X\in \overline{cog}((U\oplus R)_R)$. We may assume $X\leq E(U_R)$. Then there exists a finitely $(U\oplus R)_R$-generated module Y_R such that $X\leq Y$. There also exists a homomorphism $f: Y\rightarrow E(U_R)$ such that $f\mid X$ is the inclusion. Then since $f(Y)$ is finitely $(U\oplus R)_R$-generated, it is cogenerated by

U_R. Hence X is also cogenerated by U_R. Thus by Lemma 1.2 (iv), U is a QF-3″ object in $\overline{cog}((U\oplus R)_R)$.

Conversely suppose that U is a QF-3″ object in $\overline{cog}((U\oplus R)_R)$. Let M_R be a finitely $(U\oplus R)_R$-generated submodule of $E(U_R)$. Then $U\trianglelefteq(U+M)$ since $U\trianglelefteq E(U_R)$, and $(U+M)_R$ is finitely $(U\oplus R)_R$-generated. Hence by Lemma 1.2 (iv), it is cogenerated by U_R.

Now suppose U_R is a QF-3″ module. Then since $R_R\in\overline{cog}((U\oplus R)_R)$, U_R defines a Gabriel topology $\mathcal{G}$ and $\mathcal{G}$ corresponds to a hereditary torsion theory in *Mod-R*. Clearly this torsion theory is cogenerated by $E(U_R)$. Let $\mathcal{T}$ be the torsion class of this torsion theory. Let $\mathcal{E}(U_R)$ be the full subcategory of *Mod-R* such that $X\in\mathcal{E}(U_R)$ iff there exists an exact sequence $0\to X'\to X\to X''\to 0$ with $X'\in\overline{cog}((U\oplus R)_R)$ and $X''\in\mathcal{T}$. Then it is easy to see that $\mathcal{E}(U_R)$ is a strongly exact subcategory which contains $\overline{cog}((U\oplus R)_R)$ and $\mathcal{T}$. To say that M_R is $\mathcal{G}$-injective is the same as to say that M_R is divisible with respect tothe corresponding torsion theory to $\mathcal{G}$.

Lemma 2.7. *Let the notations be the same as the above. Suppose U_R is a QF-3″ module and $\mathcal{G}$-injective. Then U is a QF-3″ object in $\mathcal{E}(U_R)$.*

Proof. Let $U\trianglelefteq X$ be an essential extension with $X\in\mathcal{E}(U_R)$. Let $0\to X'\to X\to X''\to 0$ be exact with $X'\in\overline{cog}((U\oplus R)_R)$ and $X''\in\mathcal{E}$. Let t be the same as Lemma 1.2. We may assume $X'\leq X$ and $X''=X/X'$. Since $X'+U\in\overline{cog}((U\oplus R)_R)$ and $X/(X'+U)\in\mathcal{T}$, we may assume$U\leq X'$ by replacing $X'+U$ by X'. Hence $X'\trianglelefteq X$. Suppose $t(X)\neq 0$. Then $t(X)\cap X'\neq 0$. SinceX'_R is cogenerated by U_R by Lemma 1.2 (v), there exists a homomorphism $f: X'\to U$ such that $f(t(X)\cap X')\neq 0$. Then since U_R is $\mathcal{G}$-injective, there exists a homomor phism $g: X\to U$ such that $g\mid X'=f$. This implies $g(t(X))\neq 0$. This contradicts to the defi nition of $t(X)$.Hence $t(X)=0$. Thus by Lemma 1.2 (iv), U is QF-3″ in $\mathcal{E}(U_R)$.

Theorem 2.8. *A bimodule ${}_SU_R$ defines a localized Morita duality if and only if the following conditions are satisfied.*

(i) U_R and ${}_SU$ are QF-3″ modules.

(ii) U_R and ${}_SU$ are divisible with respect to the hereditary torsion theories cogenerated by $E(U_R)$ and $E({}_SU)$, respectively.

(iii) The canonical homomorphisms $R\to End_S(U)$ and $S\to End_R(U)$ are the localizations with respect to the torsion theories defined in (ii).

Proof. Only if part has already been proved in Theorem 1.3. So suppose ${}_SU_R$ satisfies the conditions. Let the notations be the same as Lemma 2.7. We show that ${}_SU_R$ defines a localized Morita duality between $\mathcal{E}(U_R)$ and $\mathcal{E}({}_SU)$. First we show that $X^*=Hom_R(X, U) \in \mathcal{E}({}_SU)$ if $X \in \overline{cog}((U\oplus R)_R)$. By assumption there exists Y_R such that $X \leq Y$ and Y_R is finitely $(U\oplus R)_R$-generated. By Lemma 2.3, we have an exact sequence $Y^* \to X^* \to T \to 0$ with $T \in \mathcal{T}'$, where $\mathcal{T}'$ is the torsion class of the hereditary torsion theory cogenerated by $E({}_SU)$. Then since ${}_SY^*$ is finitely ${}_S(U\oplus S)$-cogenerated, $Im(Y^* \to X^*) \in \overline{cog}({}_S(U\oplus S))$. Hence $X^* \in \mathcal{E}({}_SU)$. Next let $X \in \mathcal{E}(U_R)$ and let $0 \to X' \to X \to X'' \to 0$ be exact with $X' \in \overline{cog}((U\oplus R)_R)$ and $X'' \in \mathcal{T}$. Then $0 \to X^* \to X'^*$ is exact and $X'^* \in \mathcal{E}({}_SU)$ as we just have seen. Hence $X^* \in \mathcal{E}({}_SU)$. Since the canonical homomorphisms $\eta_U : U \simeq U^{**}$ and $\eta_R : R \simeq End_S(U)$ are the localizations, η_X is also a localization for any $X \in \overline{cog}((U\oplus R)_R)$ by Theorem 2.1. Now let $0 \to X' \to X \to X'' \to 0$ be exact with $X' \in \overline{cog}((U\oplus R)_R)$ and $X'' \in \mathcal{T}$. Then since the double U-dual functors $(\ )^{**}$ are left exact, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow \eta_{X'} & & \downarrow \eta_X & & & & \\
0 & \longrightarrow & X'^{**} & \longrightarrow & X^{**} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Coker\ \eta_{X'} & \longrightarrow & Coker\ \eta_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns. Thus $Coker\,\eta_{X'} \to Coker\,\eta_X$, is an epimorphism. This implies that η_X is the localization. This completes the proof.

3. Applications

First we prove that the duality obtained in Theorem 2.8 is a Morita duality in the sense of Anh-Weigandt. A Morita dulity in the sense of Anh-Weigandt between Grothendieck categories $\mathcal{A}$ and $\mathcal{B}$ is a duality between full subcategories $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{C}' \subseteq \mathcal{B}$ such that both are strongly exact subcategories and contain generating sets (generate $\mathcal{A}$ and $\mathcal{B}$, respectively). Let ${}_SU_R$ be a bimodule which satisfies the conditions of Theorem 2.8. Let $\mathcal{C}$ and $\mathcal{C}'$ be the Giraud subcategories of $\mathcal{E}(U_R)$ and $\mathcal{E}({}_SU)$, respectively, such that the U-dual functors define a duality between $\mathcal{C}$ and $\mathcal{C}'$. We show that this duality is a Morita duality in the sense of Anh-Weigandt. Let $\mathcal{T} = \{ M \in Mod\text{-}R \mid Hom_R(M, E(U_R)) = 0 \}$ and $\mathcal{T}' = \{ N \in S\text{-}Mod \mid Hom_S(N, E({}_SU)) = 0 \}$. We identify the quotient categories $Mod\text{-}R/\mathcal{T}$ and $S\text{-}Mod/\mathcal{T}'$ with Giraud subcategories of $Mod\text{-}R$ and $S\text{-}Mod$, respectively. Thus $\mathcal{C}$

and $\mathcal{L}'$ are full subcategories of *Mod-R/$\mathcal{T}$* and *S-Mod/$\mathcal{T}'$*, respectively. We show that $\mathcal{L}$ and $\mathcal{L}'$ are strongly exact subcategories of them. Suppose $X \in \mathcal{L}$ and $X' \leq X$ in *Mod-R/$\mathcal{T}$*. Then we show that $X' \in \mathcal{L}$. There exists an exact sequence $0 \to T \to X' \to X$ in *Mod-R*with $T \in \mathcal{T}$. Put $I = Im(X' \to X)$ in *Mod-R*. Then $I \in \mathcal{E}(U_R)$. Since $0 \to T \to X' \to X$ is exact, $I^* \simeq X'^*$, *hence* $X'^{**} \simeq I^{**}$. But $\eta_I : I \to I^{**}$ is a localization. Thus $\eta_{X'} : X' \to X'^{**}$ is also a localization by the commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X'^{**} \\ \downarrow & & \downarrow \simeq \\ I & \longrightarrow & I^{**} \\ \downarrow & & \\ 0 & & \end{array}$$

On the other hand X' is torsion free divisible. Hence $X' \simeq X'^{**} \simeq I^{**} \in \mathcal{L}$. Next let $X \to X'' \to 0$ be exact in *Mod-R/$\mathcal{T}$*. Then $X \to X''$ is an epimorphism in *Mod-R*. Hence $X'' \in \mathcal{E}(U_R)$. Now $X'' \in \mathcal{L}$ since X'' is torsion free divisible by assumption. It is clear that $\mathcal{L}$ is closed under finite coproducts (= direct sums). Thus $\mathcal{L}$ is a strongly exact subcategory of *Mod-R/$\mathcal{T}$*. It is obvious that $End_S(U) \in \mathcal{L}$ is a generator of *Mod-R/$\mathcal{T}$*. Similarly $\mathcal{L}'$ also has the same property. Therefore the duality between $\mathcal{L}$ and $\mathcal{L}'$ is a Morita duality between *Mod-R/$\mathcal{T}$* and *S-Mod/$\mathcal{T}'$* in the sense of Anh-Wiegandt.

Example 3.1. *Let $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, where $\mathbf{Z}$ is the ring of rational integers, and $R = \mathbf{Z}_2^N$. Then ${}_RR_R$ defines a localized Morita duality of $\overline{mod\text{-}R} := \overline{cog}(R_R)$. Again in this case the duality is a self-duality.*

Proof. R is self-injective since R is a complete boolean ring (see [1, exercise 18.29]). Hence by Theorem 3.1, it is clear that ${}_RR_R$ satisfies the conditions of Theorem 2.1 for $\mathcal{A} = \overline{mod\text{-}R} = \mathcal{B}$

Proposition 3.2. *let U_R be a finitely generated projective module with $S = End_R(U)$ and $tr(U_R) = I$, the trace ideal of U_R. Let ${}_RV$ be injective and suppose $Ann_V I := \{ v \in V \mid Iv = 0 \} = 0$. Then the following hold.*

(i) $End_S(U \otimes_R V) \simeq End_R(V)$ canonically.

(ii) ${}_SU \otimes_R V$ is injective.

Proof. Let ${}_I\mathcal{F} = \{ {}_RX \in R\text{-}Mod \mid IX = 0 \}$ and ${}_I\mathcal{D} = \{ {}_RX \in R\text{-}Mod \mid Ann_X I = 0 \}$ then $({}_I\mathcal{F}, {}_I\mathcal{D})$ is a hereditary torsion theory in *R-Mod*. By [12, Theorem 5.3], the canonical homomorphism

$X \to Hom_S(U, U\otimes_R X)$ gives the localization with respect to $({}_1\mathcal{F}, {}_1\mathcal{D})$ for each $X\in R\text{-}Mod$. Hence ${}_R V\simeq Hom_S(U, U\otimes_R V)$ holds. Thus

$$\begin{aligned} End_S(U\otimes_R V) &= Hom_S(U\otimes_R V, U\otimes_R V) \\ &\simeq Hom_R(V, Hom_S(U, U\otimes_R V)) \\ &\simeq Hom_R(V, V)=End_R(V). \end{aligned}$$

This proves (i). (ii) is similarly proved as [28, Proposition 4.8].

COROLLARY 3.3. *Let U_R and ${}_R V$ be finitely generated projective injective with $S=End_R(U)$, $T=End_R(V)$ and the trace ideals ${}_R I_R$ and ${}_R J_R$, respectively. Moreover suppose $Ann_V I=0$ and $Ann_U J=0$. Then ${}_S U\otimes_R V_T$ defines a localized Morita duality between $\overline{cog}({}_S U)$ and $\overline{cog}(V_T)$.*

Proof. Since $V_T\simeq Hom_S(U, U\otimes_R V)_T$ and ${}_S U\otimes_R V$ is injective, $Hom_S(X, U\otimes_R V)\in\overline{cog}(V_T)$ for all $X\in\overline{cog}({}_S U)$. Similarly $Hom_T(Y, U\otimes_R V)\in\overline{cog}({}_S U)$ for all $Y\in\overline{cog}(V_T)$. Since U_R is finitely generated, ${}_S U$ is a generator. Hence $S\in\overline{cog}({}_S U)$. Moreover since ${}_R V$ is finitely generated, there is an exact sequence ${}_R R^n\to{}_R V\to 0$. Hence we have an exact sequence ${}_S U^n \to {}_S U\otimes_R V\to 0$, which implies ${}_S U\otimes_R V\in\overline{cog}({}_S U)$. Similarly $T\in\overline{cog}(V_T)$ and $U\otimes_R V_T\in\overline{cog}(V_T)$ hold. Therefore ${}_S U\otimes_R V_T$ defines a localized Morita duality between $\overline{cog}({}_S U)$ and $\overline{cog}(V_T)$.

A module U_R with $S=End_R(U)$ is called a dominant module if U_R is faithful finitely generated projective and every simple left S-module can be embedded in ${}_S U$. It is known that a minimal faithful right R-module of a right QF-3 ring is a dominant module ([24, Corollary 1.2]).

LEMMA 3.4. *Let U_R be a dominant module with $S=End_R(U)$ and the trace ideal I. Let ${}_R V$ be faithful, injective, flat and $Ann_V I=0$. Then ${}_S U\otimes_R V$ is an injective cogenerator.*

Proof. It is enough to show that every simple left S-module can be embedded in ${}_S U\otimes_R V$. Let M be any maximal left ideal of S. Since U_R is dominant there exists $u\in U$ such that $\ell_S(u)$, the left annihilator of u, is equal to M. We show that there exists $v\in V$ such that $\ell_S(u\otimes v)=M$. Since U_R is finitely generated projective, there exists a free right R-module F with a basis $\{x_1, \dots, x_n\}$ and a right R-module U' such that $F=U\otimes U'$. Suppose $u\otimes V=0$ in $U\otimes_R V$. Then since $F\otimes_R V\simeq(U\otimes_R V)\oplus(U'\otimes_R V)$, $u=x_1r_1+\cdots+x_nr_n$ implies $(x_1r_1+\cdots+x_nr_n)\otimes V=0$ in $F\otimes_R V$. On the other hand $F\otimes_R V\simeq\bigotimes_{i=1}^{n} V$ by $(\sum x_i a_i)\otimes v\leftrightarrow$

$(a_i v)_{1\leq i\leq n}$. Thus $(\sum x_i r_i)\otimes V=0$ implies $r_i V=0$ for all i. Then $r_i=0$ since ${}_R V$ is faithful. Thus $u=0$, a contradiction. Hence there exists $v\in V$ such that $u\otimes v\neq 0$ in $U\otimes_R V$. Since $\ell_S(u)$ is maximal and $\ell_S(u)\subseteq \ell_S(u\otimes v)\neq S$, $\ell_S(u)=\ell_S(u\otimes v)$ holds. This proves that $S/M \simeq S(u\otimes v) \hookrightarrow {}_S U\otimes_R V$.

Corollary 3.5. *Let U_R and ${}_R V$ be dominant modules with $S = End_R(U)$ and $T = End_R(V)$. Moreover suppose U_R and ${}_R V$ are injective. Then ${}_S U\otimes_R V_T$ defines a Motita duality.*

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References

1. F.W. Anderson and K.R. Fuller "Rings and Categories of Modules," Springer GTM 13 (Berlin, Heidelberg, New York), 1974.
2. P.N. Anh and R. Wiegandt, *Morita duality for Grothendieck categories, preprint.*
3. R.R. Colby, *Tilting modules, dominant dimension and exactness of dual functors*, Tsukuba J. Math.. **12** (1988), 441-449.
4. R.R. Colby and K.R. Fuller, *Exactness of double dual and Morita duality for Grothendieck categories*, J. Algebra **82** (1983), 516-538.
5. R.R. Colby and K.R. Fuller, *On rings whose double dual functors preserve monomorphisms*, J. Pure Appl. Algebra **28** (1983), 223-234.
6. R.R. Colby and K.R. Fuller, *QF-3' rings and Morita duality*, Tsukuba J. Math. **3** (1984), 183-188.
7. K.R. Fuller, "Artinian Rings," Universidad Murcia, 1989.
8. J.S. Gómez Pardo and P.A. Guil Asensio, *Morita dualities associated with the R-dual functors, preprint.*
9. M. Hoshino, *Localization in abelian categories and double dual functors*, Arch. Math. **57** (1991), 245-351.
10. T. Kato, *Self-injective rings*, Tohoku Math. **19 (2)** (1967), 485-495.
11. T. Kato, *Dominamt modules*, J. Algebra **14** (1970), 341-349.
12. T. Kato, *Duality between colocalization and localization*, J. Algebra **55** (1978), 351-374.
13. J. Lambek, "Lectures on Rings and Modules," Chelsea, 1966/1986.
14. K. Masaike, *On quotient rings and torsionless modules*, Sci. Rep. Tokyo Kyoiku Daigaku **A 11** (1971), 26-30.
15. K. Masaike, *Semiprimary QF-3 quotient rings*, Comm. Algebra **11** (1983), 377-389.
16. K. Masaike, *Duality of quotient modules and a characterization of reflexive modules*, J. Pure Appl. Algebra **28** (1983), 267-277.

17. B. Mitchel, "Theory of Categories," Academic Press (New York), 1965.
18. K. Morita, *Duality of modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku **A 6** (1958), 83-142.
19. K. Morita, *Localization in categories of modules I*, Math. Z. **114** (1970), 121-144.
20. B.J. Müller, *Linear compactness and Morita duality*, J. Algebra **16** (1970), 60-66.
21. K. Ohtake, *Equivalence between colocalization and localization in abelian categories with applications to the theory of modules*, J. Algebra **79** (1982), 169-205.
22. K. Ohtake, *A generalization of Kato's theorem on Morita duality*, J. Pure Appl. Algebra **17** (1980), 323-332.
23. N. Popescu, "Abelian Categories with Applications to Rings and Modules," Academic Press (New York), 1973.
24. E.A. Rutter Jr., *Dominant modules and finite localizations*, Tohoku Math. J. **27** (1975), 225-239.
25. F.L. Sandmierski, *Linearly compact modules and local Morita duality*, in "Ring Theory (R. Gordon, ed.)," Academic Press (New York), 1972, pp. 333-346.
26. B. Stensrröm, "Rings of Quotients," Springer (Berlin, Heidelberg, New York), 1975.
27. H. Tachikawa, *On left QF-3 rings*, Pacific J. Math. **32** (1970), 255-268.
28. H. Tachikawa, "Quasi-Frobenius Rings and Generalizations," Springer **LMN** 351 (Berlin, Heidelberg, New York), 1972.
29. H. Tachikawa and K. Ohtake, *Colocalization and localization in abelian categories*, J. Algebra **56** (1979), 1-23.