NOTE BOOK

Calculus

II

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6

$\int R(\sin x, \cos x, \tan x, \cot x)\,dx$, R: rational f.

[illegible] rational f. ノ integrat. [illegible]

$\tan\frac{x}{2} = t$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos^2\frac{x}{2}\cdot 2\tan\frac{x}{2} = \frac{2t}{1+t^2}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1-\tan^2\frac{x}{2}\right) = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$

$I = \int R(\sin x, \cos x, \tan x, \cot x)\,dx$

$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}, \frac{1-t^2}{2t}\right)\frac{2\,dt}{1+t^2}$

[illegible] simple [illegible]

$\int R(\sin^2 x, \cos x)\sin x\,dx$

$\cos x = t$,

$\int R(\cos^2 x, \sin x)\cos x\,dx$

$\sin x = t$

$x^2+y^2=r^2$ $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x = r\cos\theta$ $\quad x = a\cos\theta$ $\quad x = a\sec\theta$

$y = r\sin\theta$ $\quad y = b\sin\theta$ $\quad y = b\tan\theta$

ヲ θ ノ rat. f. [illegible] parameter ノ rat. f. [illegible] curve ヲ rational curve 又 Unicursal c. [illegible]

[illegible] reduction ノ method [illegible]

$$J_n = \int \sin^n x\, dx$$

$$\frac{d}{dx}\left(-\sin^{n-1} x \cos x\right) = \sin^n x - (n-1)\sin^{n-2} x \cos^2 x$$
$$= n \sin^n x - (n-1)\sin^{n-2} x$$

$$-\sin^{n-1} x \cos x = n J_n - (n-1) J_{n-2}$$

$$J_n = \frac{n-1}{n} J_{n-2} - \frac{\sin^{n-1} x \cos x}{n}$$

$$\frac{n-1}{n} \Big| \; J_{n-2} = \frac{n-3}{n-2} J_{n-4} - \frac{1}{n-2} \sin^{n-3} x \cos x$$

$$\frac{n-3}{n-2} \Big| \; J_{n-4} = \frac{n-5}{n-4} J_{n-6} - \frac{1}{n-4} \sin^{n-5} x \cos x$$

n even $J_2 = \frac{1}{2} J_0 - \frac{1}{2} \sin x \cos x$ $\qquad \frac{(n-1)(n-3)\cdots 3}{n(n-2)\cdots 4}$

n odd $J_3 = \frac{2}{3} J_1 - \frac{1}{3} \sin^2 x \cos x$ $\qquad \frac{(n-1)(n-3)\cdots 4}{n(n-2)\cdots 5}$

$$J_n = -\cos x \left\{ \frac{\sin^{n-1} x}{n} + \frac{n-1}{n(n-2)} \sin^{n-3} x + \frac{(n-1)(n-3)}{(n)(n-2)(n-4)} \sin^{n-5} x + \cdots + \frac{(n-1)(n-3)\cdots 3.1}{n(n-2)\cdots 4.2} \sin x + \frac{(n-1)\cdots 3.1}{n(n-1)\cdots 4.2} J_0 \right\}$$ n even

$$J_0 = x.$$

or $$J_n = -\cos x \left\{ \frac{\sin^{n-1} x}{n} + \frac{n-1}{n(n-4)} \sin^{n-3} x + \cdots + \frac{(n-1)(n-3)\cdots 4}{n(n-2)\cdots 5.3} \sin^2 x \right\} + \frac{(n-1)(n-5)\cdots 4.2}{n(n-2)\cdots 5.3.1} J_1$$

$$J_1 = \int \sin x\, dx = -\cos x \qquad n, \text{odd}$$

$$J_n = \int \frac{dx}{\sin^n x}$$

$$\frac{d}{dx}\left(-\frac{\cos x}{\sin^{n+1} x}\right) = \frac{1}{\sin^n x} + (n+1)\frac{\cos^2 x}{\sin^{n+2} x}$$
$$= \frac{n+1}{\sin^{n+2} x} - \frac{n}{\sin^n x}$$

$$-\frac{\cos x}{\sin^{n+1} x} = (n+1) J_{n+2} - n J_n$$

$$J_n = -\cos x \left\{ \frac{1}{n-1} \frac{1}{\sin^{n-1} x} + \frac{n-2}{(n-1)(n-3)} \frac{1}{\sin^{n-3} x} + \frac{(n-2)(n-4)}{(n-1)(n-3)(n-5)} \frac{1}{\sin^{n-5} x} + \cdots + \frac{(n-2)(n-4)\cdots 4}{(n-1)(n-3)\cdots 3} \frac{1}{\sin^3 x} \right\} - \frac{(n-2)(n-4)\cdots 4.2}{(n-1)(n-3)\cdots 5.3} \cot x$$

n, even.

or $$J_n = -\cos x \left\{ \frac{1}{n-1} \frac{1}{\sin^{n-1} x} + \frac{n-2}{(n-1)(n-3)} \frac{1}{\sin^{n-3} x} + \cdots + \frac{(n-2)(n-4)\cdots 5.3}{(n-1)(n-3)\cdots 4.2} \frac{1}{\sin^2 x} \right\} + \frac{(n-2)(n-4)\cdots 3.1}{(n-1)(n-3)\cdots 4.2} \log\left(\tan\frac{x}{2}\right)$$

n, odd.

$$\int \sin^n x \cos^m x\, dx. \quad (m, n \text{ pos. or neg. integers})$$

1. n odd, $= 2\nu + 1$ $\qquad \cos x = t$
$$J = -\int (1-t^2)^\nu t^m\, dt$$

2. m, odd $= 2\mu + 1$ $\qquad \sin x = t$
$$J = \int t^n (1-t^2)^\mu\, dt.$$

3. $n = 2\nu$, $m = 2\mu$, $\qquad \tan x = t$
$$J_n = \int \frac{t^{2\nu}\, dt}{(1+t^2)^{\mu+\nu+1}}$$

$$\int \frac{dx}{\sin^2 x} = -\cot x \qquad \int \frac{dx}{\sin x \cos x} = \log(\tan x)$$

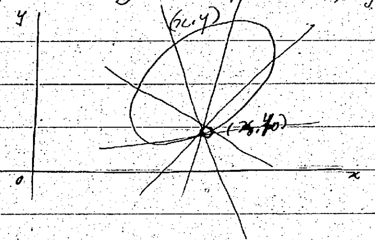
$$\int \frac{dx}{\cos^2 x} = \tan x \qquad \int \frac{dx}{\sin x} = \log\left(\tan \frac{x}{2}\right)$$

$$\int \frac{dx}{\cos x} = \log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

Integration of quadratic irrational functions

$\int R(x, \sqrt{ax^2+bx+c})\,dx$ R, rat. funct.

[illegible] $\int R(x, y)\,dx$, $y^2 = ax^2+bx+c$.



$y^2 = ax^2+bx+c$ [illegible] conic

$y - y_0 = t(x-x_0)$

t [illegible] parameter [illegible] $(x_0 y_0)$ [illegible] line-pencil [illegible] conic [illegible] intersection [illegible] $(x_0 y_0)$ [illegible]

$y = y_0 + t(x-x_0)$

$y_0^2 + 2y_0(x-x_0)t + t^2(x-x_0)^2 = ax^2+bx+c$

$y_0^2 = ax_0^2+bx_0+c$

$(x-x_0)\{2y_0t + t^2(x-x_0) - a(x+x_0) - b\} = 0$

$x(t^2-a) = ax_0+b+x_0t^2-2y_0t$

$$x = \frac{x_0t^2-2y_0t+ax_0+b}{t^2-a}$$

$$y = \frac{-y_0t^2+(2ax_0+b)t-ay_0}{t^2-a}$$

x, y [illegible] parameter t [illegible] rational function [illegible] t [illegible] (x, y) [illegible] conic [illegible] point [illegible] x, y [illegible] parameter [illegible] conic [illegible] equation [illegible] Conic [illegible] rational Curve [illegible]

$$t = \frac{y-y_0}{x-x_0}$$

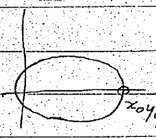
$\int R(x, y)\,dx = \int \bar{R}(t)\,dt$ [illegible] rational f. [illegible] integration [illegible]

$x_0 y_0$ [illegible] Conic [illegible] any point [illegible]

D) $ax^2+bx+c = 0$ real roots [illegible] α, β.

$y^2 = ax^2+bx+c = a(x-\alpha)(x-\beta)$

$x_0 = \alpha$, $y_0 = 0$



$t = \frac{y}{x-\alpha}$ [illegible]

$x = \frac{\alpha t^2 - a\beta}{t^2-a}$, $b = -a(\alpha+\beta)$, $c = a\alpha\beta$

$y = \frac{a(\alpha-\beta)t}{t^2-a}$

$dx = \frac{2a(\beta-\alpha)t}{(t^2-a)^2}dt$ $\frac{dx}{y} = \frac{-2}{t^2-a}dt$

$\int R(x, \sqrt{x^2-a^2})\,dx$ $a=1$, $\alpha = a$, $\beta = -a$

$t = \frac{y}{x-a}$, $x = \frac{a(t^2+1)}{t^2-1}$, $y = \frac{2at}{t^2-1}$, $\frac{dx}{y} = \frac{-2dt}{t^2-1}$

$\int \bar{R}(t)\,dt$ [illegible]

$\int R(x, \sqrt{a^2 - x^2})\,dx \qquad a = -1,$

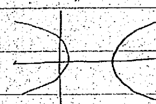
$t = \dfrac{y}{x-a},\quad x = \dfrac{-a(t^2-1)}{t^2+1},\quad y = \dfrac{2at}{t^2+1},\quad \dfrac{dx}{y} = \dfrac{-2dt}{t^2+1}$

$z.\ \beta.$

$$\int \frac{x^m dx}{\sqrt{x^2-a^2}} = -2a^m \int \frac{(t^2+1)^m dt}{(t^2-1)^{m+1}}$$

$$\int \frac{x^m dx}{\sqrt{a^2-x^2}} = 2(-1)^{m-1} a^m \int \frac{(t^2-1)^m dt}{(t^2+1)^{m+1}}$$

2). $ax^2+bx+c=0.$ complex roots



$y^2 = ax^2+bx+c = a\{(x-\alpha)^2+\beta^2\} \qquad a>0.$

$x_0 = \alpha,\quad y_0 = \sqrt{a}\,\beta$

$t = \dfrac{y-\sqrt{a}\beta}{x-\alpha},\quad x = \dfrac{a\alpha + 2\sqrt{a}\beta t - \alpha t^2}{a-t^2}$

$y = \dfrac{\sqrt{a}\beta(a+t^2)}{a-t^2},\quad \dfrac{dx}{y} = \dfrac{-2dt}{t^2-a}$

$\int R(x,y)\,dx = \int \bar{R}(t)\,dt$

$y^2 = x^2+a^2$

$t = \dfrac{y-a}{x},\quad x = \dfrac{2at}{1-t^2},\quad y = \dfrac{a(1+t^2)}{1-t^2}$

$\dfrac{dx}{y} = \dfrac{2dt}{1-t^2}$

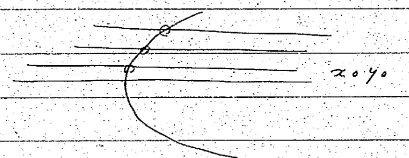
$$\int \frac{x^m dx}{\sqrt{x^2+a^2}} = 2^{m+1} a^m \int \frac{t^m dt}{(1-t^2)^{m+1}}$$

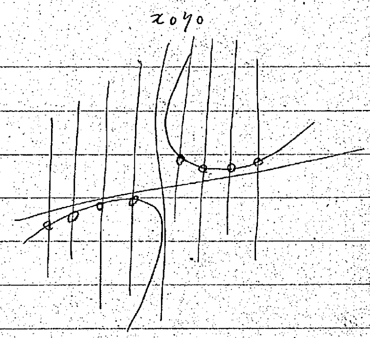
substitut ∴ [illegible]

$y^2 = ax^2+bx+c$

$c>0 \quad x_0 = 0,\ y_0 = \sqrt{c},\ t = \dfrac{y-\sqrt{c}}{x}$

$a \quad a \gtrless 0$ Hyperbola, Parabola $t = y+\sqrt{a}\,x \quad x_0, y_0$, point at ∞





Ex. 1. $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}} \qquad y^2 = ax^2+bx+c = a\left\{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right\}$

$= a\left\{\left(x+\dfrac{b}{2a}\right)^2 - \dfrac{D}{4a^2}\right\}$

$D = b^2-4ac$, Discriminant

$D>0 \qquad \alpha = \dfrac{-b+\sqrt{D}}{2a},\quad \beta = \dfrac{-b-\sqrt{D}}{2a}$

$\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}} = 2\int \frac{dt}{t^2-a} = \frac{1}{\sqrt{a}} \log \frac{t+\sqrt{a}}{t-\sqrt{a}} \qquad a>0$

or $-\dfrac{2}{\sqrt{|a|}} \arctan \dfrac{t}{\sqrt{|a|}},\quad a<0$

$t = \dfrac{y}{x-\alpha},\quad J = \dfrac{1}{\sqrt{a}} \log\left(\dfrac{y+\sqrt{a}(x-\alpha)}{y-\sqrt{a}(x-\alpha)}\right) \qquad y = \sqrt{a}\sqrt{(x-\alpha)(x-\beta)}$

$= \dfrac{1}{\sqrt{a}} \log\left(\dfrac{\sqrt{x-\beta}+\sqrt{x-\alpha}}{\sqrt{x-\beta}-\sqrt{x-\alpha}}\right)$

$= \dfrac{1}{\sqrt{a}} \log \dfrac{(\sqrt{x-\beta}+\sqrt{x-\alpha})^2}{\alpha-\beta} = \dfrac{2}{\sqrt{a}} \log(\sqrt{x-\alpha}+\sqrt{x-\beta}) + C$

or $J = -\dfrac{2}{\sqrt{|a|}} \arctan \sqrt{\dfrac{\beta-x}{x-\alpha}}$

$= \dfrac{-2}{\sqrt{|a|}}\ \mathrm{arc}\,\cancel{c}\ \sqrt{\dfrac{\beta-x}{\beta-\alpha}} + c$

$= \dfrac{2}{\sqrt{|a|}} \arccos \sqrt{\dfrac{\beta-x}{\beta-\alpha}} + c$

$D<0,\ a>0;\quad J = 2\int \frac{dt}{a-t^2} = \frac{1}{\sqrt{a}} \log \frac{t+\sqrt{a}}{t-\sqrt{a}}$

$$= \frac{1}{\sqrt{a}} \log \frac{y-\sqrt{a}\,\beta+\sqrt{a}(x-\alpha)}{y-\sqrt{a}\,\beta-\sqrt{a}(x-\alpha)}$$

$$= \frac{1}{\sqrt{a}} \log \frac{(y+\sqrt{a}\,\beta+\sqrt{a}(x-\alpha))^2}{-2\sqrt{a}\,\beta y}$$

$D=0 \qquad y = \sqrt{a}\left(x+\frac{b}{2a}\right)$

$$\int \frac{dx}{y} = \frac{1}{\sqrt{a}} \int \frac{dx}{x+\frac{b}{2a}}$$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)$$

$$\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = 2\arcsin\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\int \frac{dx}{\sqrt{(x-\alpha)^2+\beta^2}} = \log\left(\sqrt{(x-\alpha)^2+\beta^2}+x-\alpha\right)$$

Ex 2. $\displaystyle\int \frac{dx}{(x-e)\sqrt{ax^2+bx+c}}$

$x_0 = e,\ y_0 = \sqrt{ae^2+be+c} \qquad ae^2+be+c \geqq 0.$

$$t = \frac{y-y_0}{x-x_0}$$

$$x-x_0 = \frac{2ax_0+b-2y_0t}{t^2-a},\quad \frac{dx}{y} = \frac{-2dt}{t^2-a}$$

$$\int \frac{dx}{(x-x_0)y} = -2\int \frac{dt}{(2ax_0+b)-2y_0t} = \frac{1}{y_0}\log(2y_0t-2ax_0-b)$$

Ex. 3 $\displaystyle\int R(x, \sqrt{ax+b}, \sqrt{cx+d})\,dx$

$ax+b = t^2$

$$\int R\left(\frac{t^2-b}{a}, t, \sqrt{At^2+Bt+C}\right)\frac{2t\,dt}{a}$$

Ex. 4. $\displaystyle\int \frac{(mx+n)\,dx}{(\alpha x^2+\beta x+\gamma)\sqrt{ax^2+bx+c}}$ Czuber [illegible] II. S. 72.

Elliptic integrals.

$\displaystyle\int R(x,y)\,dx, \qquad y^2 = a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$

$a_0 = 0$ ナラバ [illegible]

[illegible]

$$J = \int \frac{dx}{y} = \int \frac{dx}{\sqrt{a_0x^4+\cdots+a_4}}$$

$a_0 = 0$ ナラバ $x = \frac{1}{z}$. $\quad dx = -\frac{dz}{z^2}$, $a_0x^4+\cdots+a_4$

$$J = -\int \frac{dz}{\sqrt{a_4z^4+a_3z^3+a_2z^2+a_1z}} \qquad = a_1x^3+\cdots+a_4 = \frac{a_4z^3+a_3z^2+a_2z+a_1}{z^3}$$

$\therefore\ a_4 \neq 0$ ナラバ [illegible]

$a_0 = a_4 = 0,\quad x = z^2$

$$J = 2\int \frac{dz}{\sqrt{a_1z^4+a_2z^2+a_1}}$$

[illegible]

Variable ノ change ニテ $A_0t^4+A_2t^2+A_4$ [illegible]

$$x = \frac{\alpha t+\beta}{\gamma t+\delta} \qquad (\alpha\delta-\beta\gamma \neq 0)$$

$$dx = \frac{(\alpha\delta-\beta\gamma)\,dt}{(\gamma t+\delta)^2}$$

$a_0(x^2+ax+b)(x^2+cx+d)$

$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 = \frac{a_0}{(\gamma t+\delta)^4}\left\{(\alpha t+\beta)^2+a(\alpha t+\beta)(\gamma t+\delta)+(\gamma t+\delta)^2\right\}\left\{(\alpha t+\beta)^2+c(\alpha t+\beta)(\gamma t+\delta)+d(\gamma t+\delta)^2\right\}$

$$J = \frac{(\alpha\delta - \beta\gamma)\,dt}{\sqrt{a_0(\quad)(\quad)}}$$

$A_0 t^4 + A_2 t^2 + A_4$ [illegible]

$2\alpha\beta + a(\alpha\delta + \beta\gamma) + 2b\gamma\delta = 0$

$2\alpha\beta + c(\alpha\delta + \beta\gamma) + 2d\gamma\delta = 0$

$\alpha\beta\gamma\delta$ [illegible] $\gamma = \delta = 1$ [illegible]

$2\alpha\beta + a(\alpha+\beta) + 2b = 0$

$2\alpha\beta + c(\alpha+\beta) + 2d = 0$

[illegible] $(u-\alpha)(u-\beta) = 0$

$u^2 - (\alpha+\beta)u + \alpha\beta = 0$

$\alpha + \beta = -\frac{2(b-d)}{a-c} \qquad \alpha\beta = \frac{bc-ad}{a-c}$

$(a-c)u^2 + 2(b-d)u + bc - ad = 0 \qquad \alpha, \beta$ roots

α, β real [illegible] $(b-d)^2 - (a-c)(bc-ad) \geqq 0$

4. roots x_1, x_2, x_3, x_4

$(x-x_1)(x-x_2) = x^2 + ax + b \qquad a = -(x_1+x_2)$

$\qquad b = x_1 x_2$

$(x-x_3)(x-x_4) = x^2 + cx + d \qquad c = -(x_3+x_4)$

$\qquad d = x_3 x_4$

$b - d = x_1 x_2 - x_3 x_4$

$a - c = -(\overline{x_1+x_2} - \overline{x_3+x_4})$

$bc - ad = -x_1 x_2(x_3+x_4) + x_3 x_4(x_1+x_2)$

$(b-d)^2 - (a-c)(bc-ad) = (x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)$

[illegible] $\geqq 0$ [illegible]

1) x_1, x_2, x_3, x_4 all real

$x_1 - - x_4$ [illegible] $(\;)(\;)(\;)(\;) \geqq 0$

2) two real, one pair, conjugate imaginary

$x_1, x_2 \qquad x_3 \; x_4 \qquad (\;)(\;)(\;)(\;) \geqq 0$

3) two pairs conj. imag.

$(x_1, x_2) \qquad (x_3, x_4) \qquad (\;)(\;)(\;)(\;) \geqq 0$

[illegible] $\geqq 0$ [illegible]

[illegible] α, β real. $\quad x = \frac{\alpha t + \beta}{1+t}$ [illegible]

$$J = C\int \frac{dt}{\sqrt{A_0 t^4 + A_1 t^2 + A_2}} = C\int \frac{dt}{\sqrt{(B_1 t^2 + B_2)(B_3 t^2 + B_4)}}$$ [illegible]

$$\int \frac{dt}{\sqrt{\pm(t^2-\lambda^2)(t^2-\mu^2)}}$$

$$\int \frac{dt}{\sqrt{\pm(t^2-\lambda^2)(t^2+\mu^2)}}$$

$$\int \frac{dt}{\sqrt{+(t^2+\lambda^2)(t^2+\mu^2)}} \qquad \int \frac{dt}{\sqrt{-(t^2+\lambda^2)(t^2+\mu^2)}}$$ [illegible]

[illegible] case [illegible] simple [illegible]

$$\int \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \qquad 0 < k^2 < 1 \text{ [illegible]}$$

[illegible] normal form [illegible]

[illegible] elliptic integral of the first kind [illegible]

$u = \sin\theta$ [illegible] $\int \frac{\cancel{\cos\theta}\, d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$ [illegible]

$k^2 = 1$ [illegible] $\int \frac{du}{1-u^2}$, logarithmic [illegible]

$k^2 = 0 \quad \int \frac{du}{\sqrt{1-u^2}} = \arcsin u$

log の inverse は exponential f.

arcsin の inverse は trigonometric f. なり

$0 < k^2 < 1$ の elliptic integral の inverse を

elliptic function といふ

$e^{x+2\pi i} = e^x \qquad 2\pi i$ の period

trigon. f. は $\qquad 2\pi$ の period

elliptic f. は二つの period を有す (2 fh. あり)

これをもつ ellip. integral を扱ふ

前の reduction の方法

1). $y = \sqrt{(t^2-\lambda^2)(t^2-\mu^2)}$

2) $y = \sqrt{-(t^2-\lambda^2)(t^2-\mu^2)}$

3) $y = \sqrt{(t^2+\lambda^2)(t^2-\mu^2)}$

4) $y = \sqrt{-(t^2+\lambda^2)(t^2-\mu^2)}$

5). $y = \sqrt{(t^2+\lambda^2)(t^2+\mu^2)}$

1) $\lambda^2 < \mu^2$ とし $\qquad \frac{\lambda^2}{\mu^2} = k^2 < 1.$

$0 < t^2 < \lambda^2: \qquad t^2 = \lambda^2 x^2 \quad y = \lambda\mu\sqrt{(1-x^2)(1-k^2x^2)}$

$$\int \frac{dt}{y} = \frac{1}{\mu}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad 0 \leq x^2 \leq 1$$

$t^2 > \mu^2: \quad t^2 = \frac{\mu^2}{x^2} \qquad y = \frac{\mu^2}{x^2}\sqrt{(1-x^2)(1-k^2x^2)}$

$$\int \frac{dt}{y} = -\frac{1}{\mu}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad 0 \leq x^2 \leq 1$$

$\lambda < t < \mu$ のとき y は imaginary となり除ける

2). $\lambda^2 < \mu^2 \qquad k^2 = \frac{\mu^2-\lambda^2}{\mu^2} < 1.$

$\lambda^2 < t^2 < \mu^2: \quad t^2 = \mu^2(1-k^2x^2) \quad 0 \leq x^2 \leq 1$

$y^2 = (\mu^2-\lambda^2)\mu^2 k^2 x^2(1-x^2)$

$dt = \mu \frac{-k^2x}{\sqrt{1-k^2x^2}}dx$

$$\int \frac{dt}{y} = -\frac{1}{k\mu}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

3). $\frac{\lambda^2}{\lambda^2+\mu^2} = k^2 < 1.$

$t^2 > \mu^2: \qquad t^2 = \frac{\mu^2}{1-x^2}, \quad (0 \leq x^2 \leq 1)$

$$\int \frac{dt}{y} = \frac{k}{\lambda}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

4) $\frac{\mu^2}{\lambda^2+\mu^2} = k^2 < 1 \qquad 0 < t^2 < \mu^2$

$t^2 = \mu^2(1-x^2) \quad (0 \leq x^2 \leq 1)$

$$\int \frac{dt}{y} = -\frac{k}{\mu}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

5). $\lambda^2 < \mu^2, \quad k^2 = \frac{\mu^2-\lambda^2}{\mu^2} < 1 \qquad t^2 = \frac{\lambda^2x^2}{1-x^2}$

$$\int \frac{dt}{y} = \frac{1}{\mu}\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

— 般に $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad 0 \leq x^2 \leq 1, \; 0 < k^2 < 1$ となる

一般に $J=\int R(x,y)\,dx$

$y^2=a_0x^4+a_1x^3+\cdots+a_4$

の場合には

$$R(x,y)=S(x)+\frac{T(x)}{y}$$

$$R(x,y)=\frac{P(x,y)}{Q(x,y)}=\frac{P_1(x)+P_2(x)y}{Q_1(x)+Q_2(x)y}\cdot\frac{Q_1-Q_2y}{Q_1-Q_2y}$$

$$=\frac{A(x)+B(x)y}{\frac{Q_1^2-y^2Q_2^2}{C(x)}} \qquad \frac{B(x)y^2}{C(x)y} \quad T(x)$$

$$T(x)=G(x)+\sum\frac{A_k}{(x-\alpha)^k}+\sum\frac{B_k(x)+C_k}{\{(x-\beta)^2+\gamma^2\}^k}$$

$$\int\frac{x^4dx}{y},\ \int\frac{dx}{(x-\alpha)^k y},\ \int\frac{(Bx+C)\,dx}{\{(x-\beta)^2+\gamma^2\}^k y}$$ ~~and~~ に帰し

dx :

$$\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\ \int\frac{(1-k^2x^2)dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\ \int\frac{dx}{(x^2-\alpha)\sqrt{(\ )(\ )}}$$

$0<k^2<1$

elliptic int. of the first kind. $0\leqq x^2\leqq 1$ [illegible]

" 2nd kind

" 3rd kind という [illegible]

$\left(x=\sin\theta.\ \overset{\text{1st.}}{\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}},\ \overset{\text{2nd}}{\int\sqrt{1-k^2\sin^2\theta}\,d\theta},\right)$

the integral は elementary transcendental f. にならない

elliptic integral は [illegible] pendulum [illegible]

[illegible] 計算には k が 0 から 1 に [illegible] func.

[illegible] 計算 [illegible]

[illegible] power series に expand して計算し

表 計算 [illegible] : Jahnke-Emde, Funktionentafeln

にある $\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, $\int_0^\theta\sqrt{1-k^2\sin^2\theta}\,d\theta$

3次以上 Hyperelliptic integral という

[illegible] coefficient が special のとき [illegible]

funct. に reduce される [illegible] Pseudo-elliptic int.

[illegible] 例 : $a_0=a_4,\ a_1=a_3$ [illegible]

[illegible]

Ex. 1. $J=\int\sqrt{1+\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \qquad x=a\cos\theta,\ y=b\sin\theta\ (a>b)$

[illegible] $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ [illegible] x, y [illegible]

$\int ds=\int\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$ = ellipse の curve length

[illegible]

$$J=\int\sqrt{1+a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

$$=\int\sqrt{(1+b^2)+(a^2-b^2)\sin^2\theta}\,d\theta$$

$$=\sqrt{1+b^2}\int\sqrt{1+\frac{a^2-b^2}{1+b^2}\sin^2\theta}\,d\theta$$

elliptic integral の normal form に reduce する

$$\frac{\pi}{2}-\theta=\varphi$$

$$J=-\int\sqrt{1+a^2\cos^2\varphi+b^2\sin^2\varphi}\,d\varphi$$

$$=-\int\sqrt{(1+a^2)-(a^2-b^2)\sin^2\varphi}\,d\varphi$$

$$=-\sqrt{1+a^2}\int\sqrt{1-\frac{a^2-b^2}{a^2+1}\sin^2\varphi}\,d\varphi \qquad \frac{a^2-b^2}{a^2+1}=k^2<1$$

$J=-\sqrt{1+a^2}\int\sqrt{1-k^2\sin^2\varphi}\,d\varphi$ normal ellipt. int. of the 2nd kind

$$=-\sqrt{1+a^2}\int\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx=-\sqrt{1+a^2}\int\frac{(1-k^2x^2)dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

elliptic integral ニ [illegible] ellipse ノ curve length ヲ [illegible]

Ex. 2 $J=\int \frac{d\lambda}{\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)(c^2-\lambda^2)}}$ $(a^2>b^2>c^2)$

λ ハ elliptic coordinates ナリ

$\lambda=\frac{1}{\mu}$

$$J=-\int\frac{d\mu}{\sqrt{\mu(a^2\mu-1)(b^2\mu-1)(c^2\mu-1)}}$$

$\frac{1}{a^2}=A^2,\ \frac{1}{b^2}=B^2,\ \frac{1}{c^2}=C^2\quad (A^2<B^2<C^2)$

$$=-ABC\int\frac{d\mu}{\sqrt{\underset{x_4}{\mu}\underset{x_3}{(\mu-A^2)}\underset{x_2}{(\mu-B^2)}\underset{x_1}{(\mu-C^2)}}}$$

μ ノ root ハ $0,\ A^2,\ B^2,\ C^2$

$L(\mu)=(\mu-B^2)(\mu-C^2)=\mu^2-(B^2+C^2)\mu+B^2C^2$

$M(\mu)=\mu(\mu-A^2)=\mu^2-A^2\mu$

$\mu=\frac{\alpha t+\beta}{t+1}$

$(t+1)^2L(\mu)=[(\alpha-B^2)t+(\beta-B^2)][(\alpha-C^2)t+(\beta-C^2)]$

$(t^2+1)M(\mu)=(\alpha t+\beta)[(\alpha-A^2)t+(\beta-A^2)]$

$(\ )t^2+(\ )t+(\ )$

$(\ )\left(t+\frac{\beta-B^2}{\alpha-B^2}\right)$

$(\ )\left(t+\frac{\beta-C^2}{\alpha-C^2}\right)$

$\xi=-\frac{\beta-B^2}{\alpha-B^2}=+\frac{\beta-C^2}{\alpha-C^2}$

$\eta=\frac{\beta}{\alpha}=-\frac{\beta-A^2}{\alpha-A^2}$

$d\mu=\frac{\alpha-\beta}{(t+1)^2}dt$

$$\therefore\ J=-ABC(\alpha-\beta)\int\frac{dt}{\sqrt{\alpha(\alpha-A^2)(\alpha-B^2)(\alpha-C^2)(t^2-\xi^2)(t^2-\eta^2)}}$$

$\alpha,\ \beta$ ハ $\xi,\ \eta$ ノ equation ヨリ

$$\begin{cases}\alpha\beta-(B^2+C^2)\frac{\alpha+\beta}{2}+B^2C^2=0\\ \alpha\beta-A^2\frac{\alpha+\beta}{2}=0\end{cases}$$

$\alpha\beta=X,\quad \frac{\alpha+\beta}{2}=Y$

$X=\frac{A^2B^2C^2}{B^2+C^2-A^2},\quad Y=\frac{B^2C^2}{B^2+C^2-A^2}$

$Z^2-2YZ+X=0$ ノ root ガ $\alpha,\ \beta$

$$\begin{matrix}\alpha\\ \beta\end{matrix}=\frac{B^2C^2\pm\sqrt{B^4C^4-A^2B^2C^2(B^2+C^2-A^2)}}{B^2+C^2-A^2}$$

$\sqrt{\quad}$ ノ中ハ posit. ナリ [illegible] two roots ノ combination ヲ [illegible]

$BC\sqrt{B^2C^2-A^2(B^2+C^2)+A^4}=BC\sqrt{(B^2-A^2)(C^2-A^2)}$

$$\begin{matrix}\alpha\\ \beta\end{matrix}=\frac{B^2C^2\pm BC\sqrt{(B^2-A^2)(C^2-A^2)}}{B^2+C^2-A^2}$$

カクテ J ハ 1st, 2nd ~~form~~ case of ellipt. int. ニ [illegible] $\alpha(\alpha-A^2)(\alpha-B^2)(\alpha-C^2)$ ヲ [illegible]

[illegible] $\sqrt{\quad}$ ガ imaginary ナル [illegible]

$\alpha-B^2=\frac{BC\sqrt{\quad}+B^2C^2-B^2(B^2+C^2-A^2)}{B^2+C^2-A^2}$

$=\frac{BC\sqrt{(B^2-A^2)(C^2-A^2)}-B^2(B^2-A^2)}{〃}$

$$= \frac{B\sqrt{B^2-A^2}}{B^2 C^2 - A^2}\left(C\sqrt{C^2-A^2} - B\sqrt{B^2-A^2}\right) > 0$$

$$\alpha - C^2 = C\sqrt{C^2-A^2}\left(B\sqrt{B^2-A^2} - C\sqrt{C^2-A^2}\right) < 0$$

$$J = \frac{-ABC(\alpha-\beta)}{\sqrt{\alpha(\alpha-A^2)(\alpha-B^2)(C^2-\alpha)}}\int\frac{dt}{-(t^2-\xi^2)(t^2-\eta^2)}$$

2nd case トナル

ξ, η ノ大小ヲ比較セネバナラヌ

$$\xi^2-\eta^2 = \frac{(B^2-\beta)^2}{(B^2-\alpha)^2} - \frac{\beta^2}{\alpha^2} = \frac{B^2(\alpha-\beta)(B^2-A^2)}{\alpha^2(B^2-\alpha)^2} > 0$$

2nd case トナル

$$k^2 = \frac{\xi^2-\eta^2}{\xi^2} < 1, \qquad t^2 = \xi^2(1-k^2x^2)$$

$$J = -\frac{M}{\xi}\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$ Elliptic int. of the 1st. k

実際問題トシテハ definite int. ヲ要スルノデアル

$$J_0 = \int_{b^2}^{a^2}\frac{d\lambda}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}$$

$$\lambda = b^2 \quad \mu = \frac{1}{\lambda} = B^2, \qquad t = \frac{\beta-\mu}{\mu-\alpha} = \frac{\beta-B^2}{B^2-\alpha} = \xi$$

$$\lambda = a^2 \quad \mu = \frac{1}{\lambda} = A^2, \qquad t = \quad = \frac{\beta-A^2}{A^2-\alpha} = \eta$$

$$\mu = \frac{\alpha t+\beta}{t+1}, \qquad t = \frac{\beta-\mu}{\mu-\alpha}$$

$$\lambda = b^2, \quad x = 0$$

$$\lambda = a^2, \quad x = 1. \qquad \therefore J_0 = \frac{M}{\xi}\int_0^1\frac{dx}{\sqrt{}}$$

之ハ complete elliptic integral ニシテ [illegible] k ニ対シテ [illegible] k ハ a^2, b^2, c^2 ニヨリ [illegible] a, b, c ハ ellipsoid ノ axes ナリ

$$\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$ ヲ Legendre's normal form トナス

又 $$\int\frac{dx}{\sqrt{4x^3-g_2x-g_3}} = \int\frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$ ヲ

Weierstrass's normal form トス [illegible]

ニテ $$\int\frac{dx}{\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}$$ ハ [illegible] normal form ナリ

然レドモ [illegible] Legendre's form [illegible]

[illegible]

Legendre's normal form ノ inversion ハ elliptic funct. ナリ Weierstrass's form ノ inverse ヲ ellipt. f. ノ notation [illegible] Abel, Jacobi ノ [illegible] Weierstrass ノ elliptic f. ナリ

Ex. 3 $$J = \int\frac{\lambda\, d\lambda}{\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}} \qquad (a^2 > b^2 > c^2)$$

reduction [illegible] $\lambda = \frac{1}{\mu}$, $\mu = \frac{\alpha t+\beta}{t+1}$

$$J = M_0\int\frac{t+1}{\alpha t+\beta}\,\frac{dt}{\sqrt{-(t^2-\xi^2)(t^2-\eta^2)}}$$

$$= \frac{M}{\alpha}\int\frac{t+1}{t+\eta}\,\frac{dt}{\sqrt{-(t^2-\xi^2)(t^2-\eta^2)}}$$

$$t+1 = t+\eta+(1-\eta)$$

$$= \frac{M}{\alpha}\int \frac{dt}{\sqrt{-(t^2-\xi^2)(t^2-\eta^2)}} + \frac{(1-\eta)M}{\alpha}\int \frac{1}{t+\eta}\frac{dt}{\sqrt{\ }}$$

ellipt. of 1st kind

$$\int \frac{1}{t+\eta}\frac{dt}{\sqrt{\ }} = \int \frac{t-\eta}{t^2-\eta^2}\frac{dt}{\sqrt{\ }}$$

$$= \int \frac{t\,dt}{(t^2-\eta^2)\sqrt{\ }} \quad \int \frac{dt}{t^2-\eta^2\sqrt{\ }}$$ ellipt. int. of 3rd kind)

$$\int \frac{t\,dt}{t^2-\eta^2\sqrt{-(t^2-\xi^2)(t^2-\eta^2)}}$$, $t^2 = s$ [illegible] integral [illegible]

$$\int \frac{dt}{(t^2-\eta^2)\sqrt{\ }} = K\int \frac{1}{(\ )x^2+(\ )}\frac{dx}{\sqrt{\ }}$$ 3rd kind)

Summary

1. $\int R(x, \sqrt{ax^2+bx+c})\,dx$ elementary f.

2. $\int R(x, \sqrt{ax^3+\cdots})\,dx$, $\int R(x, \sqrt{ax^4+\cdots})\,dx$) elliptic int.

3. $n>4$ $\int R(x, \sqrt{ax^n+\cdots})\,dx$ hyperelliptic int.

inverse

1. trigon. f. exponential f.
2. ellipt. f.
3. hyperelliptic funct. [illegible]

$\int R(x, \sqrt[3]{ax^2+bx+c})\,dx$ [illegible]

[illegible] integral [illegible] inverse [illegible] Abelian funct. [illegible]

19 th. [illegible] integrat. [illegible] funct. [illegible] reduce

[illegible] integral [illegible] funct. [illegible] defin. [illegible] Abel, Jacobi [illegible] integrat. [illegible] special case [illegible] dif. equat. [illegible] funct. [illegible] introduce [illegible] funct. theory [illegible]

[illegible] 計算 [illegible] power series [illegible] expand [illegible] approximat. [illegible]

$\sqrt{ax^3}$ or ax^4 [illegible] special form [illegible] $\sqrt{ax^2+\cdots}$ [illegible] degenerat. [illegible] Pseudo-ellipt. integral [illegible]

$$\int \frac{\phi(x)\,dx}{\sqrt{a_0x^4+a_1x^3+a_2x^2+a_1x+a_0}}$$ (reciprocal eq. $x, \frac{1}{x}$ [illegible]) [illegible] $f(x)+f\left(\frac{1}{x}\right)\equiv 0$

[illegible]

$t = \frac{x-1}{x+1}$ [illegible] elementary integral [illegible]

Integration of binomial differential

$J = \int x^m(ax^n+b)^p\,dx$, ($m, n, p$ rational no.)

[illegible] p [illegible] special case [illegible]

$ax^n = bt$

$J = C\int t^q(1+t)^p\,dt \qquad q = \frac{m-n+1}{n}$, rat.

$J_{p,q} = \int t^q(1+t)^p\,dt$

1) $p>0$ $$\frac{d}{dt}\left(\frac{t^{q+1}}{q+1}(1+t)^p\right) = t^q(1+t)^p + \frac{p}{q+1}t^{q+1}(1+t)^{p-1}$$

$$\frac{t^{q+1}(1+t)^p}{q+1} = J_{p,q} + \frac{p}{q+1}J_{p-1,q+1} \quad \text{---(1)}$$

$p<0,\quad \frac{t^q(1+t)^{p+1}}{q} = J_{p+1,q-1} + \frac{p+1}{q} J_{p,q}$

$$\frac{t^q(1+t)^{p+1}}{p+1} = \frac{q}{p+1} J_{p+1,q-1} + J_{p,q} \quad \text{--- (2)}$$

(1) 又ハ (2) ニヨリ simpler form ニ reduce シ

(A). 此 reduction ノ 出来ルハ p, integer ナル時ニ限ル

$p>0$ (1) ニヨリ $J_{0,p+q}$ ニ帰ス

$p<0$ (2) ニヨリ $J_{-1,p+q+1}$ ..... J_0, ... $p+1=0$ 迄ナル

$J_{0,p+q} = \int t^{p+q}\,dt$

$J_{-1,p+q+1} = \int \frac{t^{p+q+1}}{1+t}\,dt \qquad p+q+1 = \frac{r}{s} \qquad t^{\frac{1}{s}} = u$

$J_{p,q} = \int t^q(1+t)^p\,dt$

$1+t = u \qquad J_{p,q} = \int (u-1)^q u^p\,du$

(B). q, integer, 此ハ又 p integer ニ为ス

$p = \frac{r}{s}, \qquad (1+t)^{\frac{1}{s}} = v$

$t = \frac{1}{u} \qquad J_{p,q} = -\int (1+u)^p u^{-(p+q+2)}\,du$

(C) $p+q$, integer

$p = \frac{r}{s}, \quad \left(\frac{1+t}{t}\right)^{\frac{1}{s}} = v$

此三ツノ case ハ elementary f. ニ integrate セラル

逆ニ他ノ case ハ不可能ナルコトヲ证スルコトガ出来ル

(略シオク)

Ex. $J = \int x^{\#}\sqrt[3]{1+x^3}\,dx$

$x^3 = t$

$J = \frac{1}{3}\int (1+t)^{\frac{1}{3}} t^{-\frac{1}{3}}\,dt$

$p+q = \frac{1}{3} - \frac{1}{3} = 0$ integrable.

$\frac{1+t}{t} = v^3 \qquad J = -\int \frac{v^3\,dv}{(v^3-1)^2}$ int. by part. fract.

Integration of transcendental functions

algebraic f. ノ int. ニ比シ尚ホ困ナルハ transc. f. ニ

更ニ狭イ範囲ニ限ルノデ一面ニ易キモ有ルハ少ナシ

1. $\int G(x)\sin x\,dx = -Gx\cos x + \int G'(x)\cos x\,dx$

$= -G(x)\cos x + G'(x)\sin x - \int G''(x)\sin x\,dx$

$G(x)$, rational integral f. ナラバ integrable ナリ

$G^{(n)}$ ハ constant トナルカラ

$G(x)$, rational f. G ヲ partial fract. ニ分ツ

term ハ $\frac{1}{(x-a)^m}$ ノ如キ形ナリ

$\frac{1}{(x-a)^m}$ ハ G ノ如ク考フルト reduct. ニヨリ $\frac{1}{x-a}$

ニ迄来ル $\frac{1}{(x-a)^2}$ ナラバ constant トナル

$\int \frac{\sin x\,dx}{x-a}$ 又ハ $\int \frac{\cos x\,dx}{x-a}$ ニ帰ス

此 integrate ハ出来ヌ

$Si(x) = \int \frac{\sin x\,dx}{x}$

$Ci(x) = \int \frac{\cos x\,dx}{x}$ ハ新シキ function ナリ、之ヲ sine-integral, cosine-integral

トシテ 此 integral ハ [illegible] 計算シテ 此 値ヲ表ニシテアル (Jahnke-Emde, Funktionentafel)

2 $\int G(x)e^{x}dx = G(x)e^{x} - \int G'(x)e^{x}dx$

$= e^{x}(G(x) - G'(x)\cancel{dx}) + \int G''(x)e^{x}dx$

$= \cdots$

$G(x)$, rat. integral f. integrable

$\frac{1}{x^n}$ rational f. ハ $\int \frac{e^{x}dx}{x}$ = reduce [illegible]

$\frac{e^{-x}dx}{x}$ ハ exponential integral function トシテ

$Ei(x)$ [illegible] 此ヲ表ス

$e^{-x} = t$ トス $\int \frac{t\,dt}{t\log t} = \int \frac{dt}{\log t}$ [illegible]

integral-logarithm トシテ $li(x)$ [illegible]

$Ei(x) = li(e^{-x})$

[illegible] funct. [illegible] f. ノ integral [illegible]

f. [illegible]

$\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$, Gauss's error funct. Least square [illegible]

$\frac{1}{\sqrt{2\pi}}\int_0^x \frac{\cos x\,dx}{\sqrt{x}}$ ([illegible] $2\int \cos t^2 dt$ [illegible]) $= C(x)$

$\frac{1}{\sqrt{2\pi}}\int_0^x \frac{\sin x\,dx}{\sqrt{x}}$ ($2\int \sin t^2 dt$) $= S(x)$

Fresnel's integral トス

3 $\int G'(x)\log x\,dx = G(x)\log x - \int \frac{G(x)}{x}dx$

G ガ sine, cosine, expon. ナラバ sin (cosine) integ. exp. int. トナリ G ガ rat. int. f. ナラ integrable

G ガ rational f. ナラバ integrable ナリ

[illegible] $\int x^n\log x\,dx = \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$ $(n \neq -1)$

$n = -1$, $\int \frac{\log x\,dx}{x} = \frac{(\log x)^2}{2}$

$\int \frac{1}{x}\log x\,dx = \log x\cdot\log x - \int \frac{\log x}{x}dx$

$\int x^m(\log x)^n dx$ m, n, ± integer

$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m(\log x)^{n-1}dx$ $(m \neq -1)$

n + integer ナラバ log ノ [illegible] integrable

n, − integer ナラバ 逆ニ上式ヲ用ヒテ [illegible]

$\int \frac{x^m}{\log x}dx$ [illegible]

$\int \frac{x^m dx}{\log x} = \int \frac{dt}{\log t}$ $(x^{m+1} = t)$

integral logarithm

$m = -1$, $\int \frac{(\log x)^n}{x}dx = \frac{(\log x)^{n+1}}{n+1}$ $(n \neq -1)$

$m = n = -1$, $\int \frac{dx}{x\log x} = \log(\log x)$

4. $\int \arcsin^n x\,dx = x\arcsin^n x - n\int \frac{x\arcsin^{n-1}x}{\sqrt{1-x^2}}dx$

$= x\arcsin^n x - n\left\{-\sqrt{1-x^2}\arcsin^{n-1}x + (n-1)\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\arcsin^{n-2}x\right\}$

$= x\arcsin^n x + n\sqrt{1-x^2}\arcsin^{n-1}x - n(n-1)\int \arcsin^{n-2}x\,dx$

n, + intg, n even $\int dx$ = , n odd $\int \arcsin x\, dx$ [illegible]

n, negat. int. $$\int \frac{dx}{\arcsin^n x} = \frac{-\sqrt{1-x^2}}{(n-1)\arcsin^{n-1} x} + \frac{1}{n-1}\int \frac{x}{\sqrt{1-x^2}}\frac{dx}{\arcsin^{n-1} x}$$

$$= \frac{-\sqrt{1-x^2}}{(n-1)\arcsin^{n-1} x} + \frac{1}{n-1}\left\{\frac{-x}{(n-2)\arcsin^{n-2} x} + \frac{1}{n-2}\int \frac{dx}{\arcsin^{n-2} x}\right\}$$

n + integer [illegible]

$$\int \frac{dx}{\arcsin x} \quad \int \frac{dx}{\arcsin^2 x}$$ [illegible]

$\arcsin x = t$, $x = \sin t$, $dx = \cos t\, dt$

$$\int \frac{dx}{\arcsin x} = \int \frac{\cos t\, dt}{t} \quad \text{cosine integral}$$

$$\int \frac{dx}{\arcsin^2 x} = -\frac{\sqrt{1-x^2}}{\arcsin x} + \int \frac{x}{\sqrt{1-x^2}}\frac{dx}{\arcsin x}$$

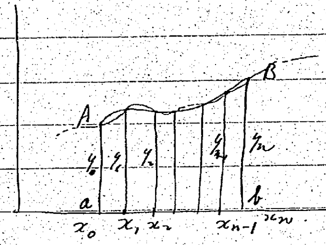
$$= -\frac{\sqrt{1-x^2}}{\arcsin x} + \int \frac{\sin t\, dt}{t} \quad \text{sine integral.}$$

[illegible] theoretical [illegible] integrat. [illegible] practical [illegible] mechanical quadrature [illegible]

Mechanical Quadrature

$f(x)$, (a, b) one valued, cont.

$J = \int_a^b f(x)\, dx$ [illegible] $\int f(x)\, dx$ [illegible]



[illegible] AabB [illegible]

x_0 y_0

x_1 y_1

x_2 y_2

— —

x_n y_n

$n+1$ points [illegible] $y = P(x)$ (P n th deg. polynomial) [illegible] curve [illegible] area [illegible]

$y = P(x)$ [illegible] Lagrange's formula of interpolat. [illegible]

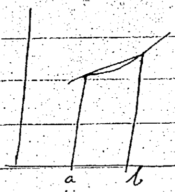
$$\varphi(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$$

$$P(x) = \sum_{k=0}^{n} \frac{y_k}{\varphi'(x_k)} \frac{\varphi(x)}{x-x_k}$$

$$J \doteq \int_a^b P(x)\, dx = \sum_{k=0}^{n} \frac{y_k}{\varphi'(x_k)} \int_a^b \frac{\varphi(x)\, dx}{x-x_k}$$

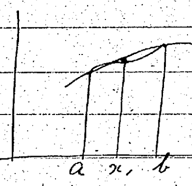
$\frac{P(x)}{x-x_k}$ n th deg. polynomial [illegible]

$n = 1$ [illegible]



polynomial straight line [illegible] [illegible] div. [illegible] [illegible] trapezoidal formula

$n = 2$

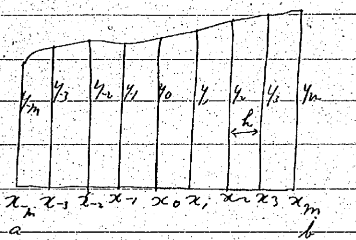


$y = P(x)$ [illegible] parabola [illegible]

[illegible] Simpson's rule [illegible]

for the sake of simplicity, $x_1, x_2, \ldots$ [illegible] Newton's Cotes' formula [illegible] Maclaurin's f. [illegible]

[illegible] m [illegible] even [illegible]



$\frac{b-a}{2m} = h$

Newton-Cotes:

$m =$ even $\begin{cases} y_0 \quad y_2 \quad y_4 \; \cdots \; y_m \\ y_{-2} \quad y_{-4} \; \cdots \; y_{-m} \end{cases}$

m odd $\begin{cases} y_1 \quad y_3 \quad y_5 \; \cdots \; y_m \\ y_{-1} \quad y_{-3} \quad y_{-5} \; \cdots \; y_{-m} \end{cases}$

[illegible] Lagrange's form [illegible] apply [illegible]

$m = 2 \qquad \frac{h}{3}(4y_0 + y_2 + y_{-2})$

$m = 3. \qquad \frac{3h}{8}(y_3 + y_{-3} + 3(y_1 + y_{-1}))$

$m = 4 \qquad \frac{2h}{45}(12y_0 + 32(y_2 + y_{-2}) + 7(y_4 + y_{-4}))$

[illegible] Newton's Cotes form. [illegible]

[illegible] y_0 ordinate [illegible] Simpson's

first) Rule [illegible] (a Simpson's one-third rule)

[illegible] Simpson's second (three-eighth) rule

m [illegible] Moore [illegible] ($m = 12$ [illegible])

Maclaurin's f.

$m =$ even $\quad y_1 \quad y_3 \; \cdots \; y_{m-1}$

$\qquad\qquad y_{-1} \quad y_{-3} \; \cdots \; y_{-(m-1)}$

$m =$ odd $\quad y_0 \quad y_2 \quad y_4 \; \cdots \; y_{m-1}$

$\qquad\qquad y_{-2} \quad y_{-4} \; \cdots \; y_{-(m-1)}$

[illegible] Moore [illegible] m [illegible] Mad. $m+1$ [illegible]

Newton Cotes [illegible] Mad. [illegible]

[illegible] exact [illegible]

[illegible] exact [illegible] Gauss's rule [illegible]

Maclaurin's rule [illegible]

$m = 3, \quad \frac{b-a}{8}\{2y_0 + 3(y_2 + y_{-2})\} \qquad h = \frac{b-a}{m}.$

$m = 4, \quad \frac{b-a}{48}\{11(y_1 + y_{-1}) + 13(y_3 + y_{-3})\}$

$m = 6, \quad \frac{h}{140}\{272y_0 + 27(y_2 + y_{-2}) + 216(y_4 + y_{-4}) + 41(y_6 + y_{-6})\}$

[illegible] modify [illegible]

$\frac{3h}{10}\{6y_0 + (y_2 + y_{-2}) + 5(y_4 + y_{-4}) + (y_6 + y_{-6})\}$ [illegible]

[illegible] Weddle's rule [illegible]

Gauss's rule

(a, b) $f(x)$, expansible into power series by Maclaurin's theorem.

$x = x_1, x_2, x_3, \cdots x_n$

$f(x) = y_1, y_2, y_3, \cdots y_n$

$J_0 = \int_a^b f(x)\,dx$

polynomial, $F(x) = \sum_{k=1}^{n} \frac{y_k}{\varphi'(x_k)} \frac{\varphi(x)}{x - x_k}$, $\varphi(x) = (x - x_1) \cdots (x - x_n)$

at most $(n-1)$ th degree

[illegible] $y = F(x)$ [illegible] Curve [illegible] $f(x)$ (x_1, y_1) $(x_2, y_2) \cdots$

(x_n, y_n) [illegible]

$J_1 = \int_a^b F(x)\,dx$ [illegible] J_0 [illegible]

$x_1, x_2 \cdots x_n$ [illegible] J_1 [illegible] J_0 [illegible]

[illegible] Gauss [illegible]

$f(x)$ [illegible] at most $(n-1)$ th degree polynomial

[illegible] $f(x)$ [illegible] $F(x)$ [illegible] Coincide [illegible] $n-1$ th deg. poly [illegible] n points [illegible] pass [illegible] only one [illegible]

[illegible] $J_0 = J_1$

~~[illegible]~~

$f(x) \equiv \varphi(x)\,Q(x) + R(x)$

$Q(x)$ [illegible] power series [illegible], $R(x)$ polynomial of at most $(n-1)$ th degree.

$J_0 = \int_a^b f(x)\,dx = \int_a^b \varphi(x)\,Q(x)\,dx + \int_a^b R(x)\,dx$

$x = x_k$ $\quad f(x_k) = \underset{0}{\varphi(x_k)}\,Q(x_k) + R(x_k) = R(x_k) = y_k$

$\therefore R(x) = F(x)$ [illegible]

$\int_a^b R(x)\,dx = \int_a^b F(x)\,dx = J_1$

$J_0 - J_1 = \int_a^b \varphi(x)\,Q(x)\,dx$ [illegible] error [illegible]

$= \int_a^b \varphi(x)(a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots)\,dx$

[illegible] $= a_0 \int_a^b \varphi(x)\,dx + a_1 \int_a^b \varphi(x)\,x\,dx + a_2 \int_a^b \varphi(x)\,x^2\,dx + \cdots + a_k \int_a^b \varphi(x)\,x^k\,dx + \cdots$

$b - a$ [illegible]

$\varphi(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$ [illegible] x_n [illegible]

[illegible] n [illegible]

$\int_a^b \varphi(x)\,dx = 0$, $\int_a^b \varphi(x)\,x\,dx = 0$, $\int_a^b \varphi(x)\,x^2\,dx = 0$, $\cdots$ $\int_a^b \varphi(x)\,x^{n-1}\,dx = 0$ [illegible]

$J_0 - J_1 = a_n \int_a^b x^n \varphi(x)\,dx + \cdots$ [illegible]

~~[illegible] integrate [illegible] x^{n+1} [illegible] $(b-a)^{n+1}$ [illegible]~~

$\underset{a}{(-h)} \quad \underset{0}{} \quad \underset{b}{(+h)}$ interval [illegible] change of variable [illegible] $\int_a^b (\ )\,dx = \int_{-h}^{h} (\ )\,dt$

$x = a$, $x = b$, $t = -h$, $t = +h$, $x = At + B$, $a = -Ah + B$, $b = Ah + B$

$\frac{a+b}{2} = B$, $\frac{b-a}{2} = A$, $x = \frac{b-a}{2h} t + \frac{b+a}{2}$. ~~[illegible]~~

~~[illegible] $\varphi(x)$ [illegible] root [illegible] x_1, $x_2, \cdots x_{n-1}, x_n$ [illegible]~~

~~[illegible] $\int_a^b x^k \varphi(x)\,dx$ [illegible]~~

$\int \varphi(x)\,dx = \varphi_1(x)$, $\int \varphi_1(x)\,dx = \varphi_2(x)$, $\cdots$ $\int \varphi_{n-1}(x)\,dx = \varphi_n(x)$

$$\int_a^b x^k\varphi(x)\,dx = \left(x^k\varphi_1(x)\right)_a^b - k\int_a^b x^{k-1}\varphi_1(x)\,dx$$
$$= \cdots - k\left(x^{k-1}\varphi_2(x)\right)_a^b + k(k-1)\int_a^b x^{k-2}\varphi_2(x)\,dx$$
$$= \left(x^k\varphi_1(x)\right)_a^b - k\left(x^{k-1}\varphi_2(x)\right)_a^b + \cdots + (-1)^k k(k-1)\cdots 2\cdot 1\left(\varphi_{k+1}(x)\right)_a^b$$

$= 0$ [illegible]

$$\left(\varphi_1(x)\right)_a^b = 0$$
$$\left(x\varphi_1(x)\right)_a^b = \left(\varphi_2(x)\right)_a^b = 0$$

$\varphi_1(x),\ \varphi_2(x), \cdots \varphi_n(x) = 0$ at $x = a, b$

[illegible]

$$\left.\begin{array}{l} \varphi_n(x) = \psi(x) \\ \varphi_{n-1}(x) = \psi'(x) \\ \varphi_{n-2}(x) = \psi''(x) \\ \cdots \\ \varphi_1(x) = \psi^{(n-1)}(x) \end{array}\right\} = 0 \quad \text{when } x = a, b$$

$$\varphi(x) = \psi^{(n)}(x)$$

at least $\psi(x) = (x-a)^n(x-b)^n\theta(x)$

$\varphi(x)$, nth $\psi(x)$ [illegible] n [illegible] diff. [illegible] nth degree [illegible]

$\theta(x)$, constant [illegible] ~~[illegible]~~ $\frac{(2n)!}{n!}$ [illegible]

$$\psi(x) = (x-a)^n(x-b)^n\frac{(2n)!}{n!}$$
$$\therefore\ \varphi(x) = \frac{(2n)!}{n!}\,\frac{d^n\left((x-a)(x-b)\right)^n}{dx^n}$$

[illegible] $\int_a^b x^k\varphi(x)\,dx = 0,\quad k = 0, 1, \cdots (n-1)$

$\int_a^b \Phi(x)\varphi(x)\,dx = 0$, $\Phi(x)$, polynomial of at most $(n-1)$th degree

$a = -1,\quad b = +1$

~~$\varphi(x)$~~ $P_n(x) = \frac{(2n)!}{n!}\,\frac{d^n(1-x^2)^n}{dx^n}$, $X_n(x)$ [illegible]

Legendre's polynomial or Zonal harmonics [illegible]

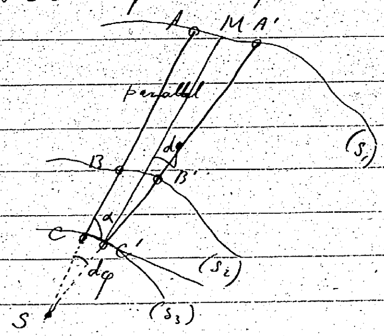
$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0,\quad m < n \text{ or } m > n \text{ [illegible] } m \neq n.$$

[illegible]

$P_n(x) = 0$ [illegible] $\varphi(x) = 0$ [illegible] roots [illegible] a [illegible] b [illegible] real, different ~~with~~ from each other [illegible] (by Rolle's theorem)

[illegible] $x_1, x_2 \cdots x_n$ [illegible] $y_1, y_2, y_3 \cdots y_n$ [illegible] mechanical quadrature [illegible] theoretically most exact [illegible] Gauss's [illegible] P_n [illegible] algebraic equat. [illegible] approximate value [illegible] (Moore's table)

Principles of Planimeters



S_1, S_2 [illegible] curve [illegible] S_2 circle or straight line [illegible]

$AB = l,\quad BC = l'$

(S_3) [illegible] $(S_1 S_2$ [illegible]$)$

$CC' = ds$

area $ACC'A' = ACC'M + MC'A'$

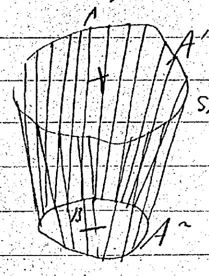
$ACC'M = (l+l')\sin\alpha\, ds$ $\qquad$ $MC'A'$ [illegible] MA' [illegible] sector

$MC'A' = \frac{1}{2}(l+l')^2 d\varphi$

$ACC'A = (l+l')\sin\alpha\, ds + \frac{1}{2}(l+l')^2 d\varphi$

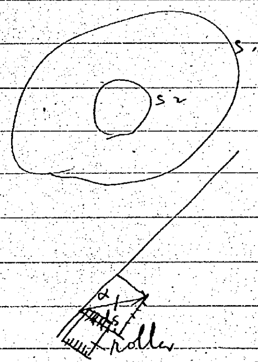
$BCC'B = l'\sin\alpha\, ds + \frac{1}{2} l'^2 d\varphi$

$ABB'A' = l\sin\alpha\, ds + \frac{1}{2}(l^2+2ll')d\varphi$



$(A_1)-(A_2) = \int l\sin\alpha\, ds + \frac{1}{2}(l^2+2ll')\int d\varphi$

φ , 0



$\varphi = 2\pi$

φ .. constant

[illegible] area [illegible] $\sin\alpha\, ds$ [illegible]

depend [illegible]

$\int\sin\alpha\, ds$ [illegible]

area difference [illegible]

(A_2) .. S_2 [illegible] instrument [illegible]

constant [illegible]

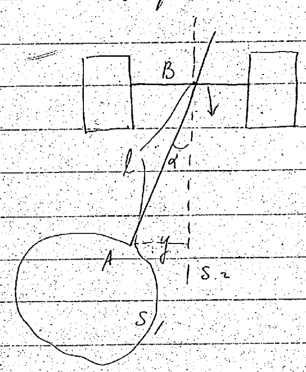
$A_1 - A_2 = l\int\sin\alpha\, ds + 0$ or $\pi(l^2+2ll')$

Swiss

Amsler's planimeter [illegible] principle [illegible]

Scheiben-planimeter [illegible] Roller pl. [illegible]

Roller planimeter $\qquad$ S_1 straight line



$ds = dx$

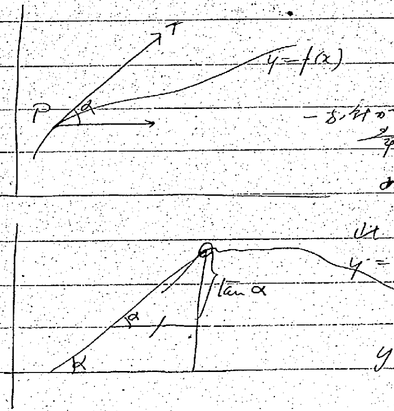
$y = l\sin\alpha$

$l\sin\alpha\, ds = y\,dx$

$\int l\sin\alpha\, ds = \int y\,dx$

[illegible] $y=f(x)$ [illegible] $\int f(x)\,dx$ [illegible] curve [illegible]

Integraph [illegible] (von Abdank)



$f'(x) = \tan\alpha$

[illegible] curve [illegible] trace [illegible]

[illegible] tangent [illegible]

[illegible] bar [illegible]

[illegible] bar [illegible]

$y=f'(x)$ [illegible]

[illegible] $y=f'(x)$ [illegible]

$y=f(x)$ [illegible] trace [illegible]

Integration of infinite series.

$a_1 + a_2 + a_3 + \cdots$

a_n は + or −

$S_n = a_1 + a_2 + \cdots + a_n$

S_n の sequence が converge するか diverge するか [illegible] inf. series が conv. 又は div. [illegible]

$$\lim_{n\to\infty} S_n = S.$$

S_n の seq. の convergence の necessary & sufficient condition は [illegible]

$\varepsilon > 0$ small at pleasure なる m, any integer

$$|S_{n+m} - S_n| < \varepsilon, \quad \text{for } n \geq n_0(\varepsilon)$$

funct. of ε

なる $n_0(\varepsilon)$ [illegible] conv. の necessary & suf. cond. [illegible]

Proof. $\sum a_n$ が conv. [illegible] sum s とす

$$\lim_{n\to\infty} S_n = S$$

ε $\quad |S_n - s| < \frac{\varepsilon}{2}$, for $n \geq n_0(\varepsilon)$

なる n_0 [illegible] (from the def. of limit)

$\therefore |S_{n+m} - S| < \frac{\varepsilon}{2}$ for $n \geq n_0(\varepsilon)$

$$|S_{n+m} - S_n| = |(S_{n+m} - S) - (S_n - S)| \leq |S_{n+m} - S| + |S_n - S|$$

$$< \varepsilon, \quad n \geq n_0$$

故に此は necessary である

次に

decreasing seq

$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m, \cdots \qquad \varepsilon_m \to 0$ とし

これに対する [illegible] n_0 [illegible]

$n_1, n_2, \cdots, n_m, \cdots$ なる seq. [illegible]

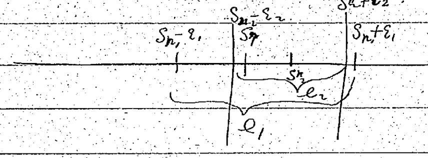
$$|S_{n+m} - S_n| < \varepsilon_k \quad \text{for } n \geq n_k \ (k, 1, 2, 3, \cdots)$$

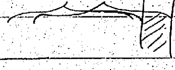
$$|S_{n_k+p} - S_{n_k}| < \varepsilon_k \qquad p \geq 0$$

$$S_{n_k} - \varepsilon_k < S_{n_k+p} < S_{n_k} + \varepsilon_k$$

S_{n_k} [illegible] $S_{n_k} \pm \varepsilon_k$ [illegible]

即ち $e_k = (S_{n_k} + \varepsilon_k,\ S_{n_k} - \varepsilon_k)$ interval [illegible]





e_k の length は $2\varepsilon_k$ である

$e_1, e_2, e_3, \cdots$ の interval [illegible]

[illegible] length [illegible]

[illegible] e_1 の中 [illegible] e_2'

[illegible] e_3' [illegible]

$e_1', e_2', e_3', \cdots \to 0$

[illegible] interval の [illegible] increase [illegible]

[illegible] decrease [illegible]

[illegible] 0 [illegible] common limit [illegible]

[illegible] S [illegible]

$$\lim S_n \longrightarrow S$$

$$|S_n - S| < \varepsilon \text{ for } n \geq N$$

[illegible] N [illegible]

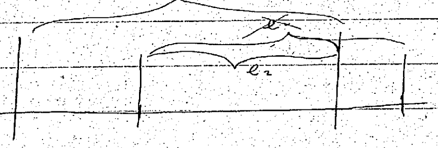
∴ S_n [illegible] e'_m [illegible]

$e'_m < \varepsilon$ [illegible]

[illegible] $|S_n - S| < \varepsilon$ [illegible]

[illegible] cond. [illegible] sufficient [illegible]

[illegible] [illegible] e_2 [illegible] e_1 [illegible]



$$S_{n_k} - \varepsilon_k < S_{n_k+p} < S_{n_k} + \varepsilon_k$$

$p \geq 1$ [illegible] S_{n_k+p} interval [illegible]

Corollary. S_{n+m}, $m = 1$

$$|a_{n+1}| < \varepsilon \text{ for } n \geq n_0$$

$$|a_n| \longrightarrow 0$$

term [illegible] 0 [illegible] tend [illegible] (necessary)

$1 - 1 + 1 - 1 - \cdots$ [illegible] converge [illegible]

$1 + \frac{1}{2} + \cdots + \frac{1}{n}$ [illegible] $\frac{1}{n}$ [illegible] 0 [illegible] tend [illegible]

divergent [illegible]

[illegible] general criterion [illegible] infinite series [illegible]

[illegible]

Alternating series [illegible] term [illegible] 0 [illegible] tend [illegible]

convergent [illegible]

$a_1 + a_2 + a_3 + \cdots$) conv.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$|a_1| + |a_2| + |a_3| + |a_4| + \cdots$) div.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

[illegible]

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ conv.

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ conv. also

[illegible] term [illegible] sign [illegible] conv. [illegible]

absolutely convergent series [illegible]

sign [illegible] conditionally conv. [illegible]

(semi-conv. [illegible] semi-conv. [illegible])

absolute value term [illegible] series [illegible] conv. [illegible]

[illegible] alternating series [illegible] conv. [illegible]

∵ $|a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+m}| < \varepsilon$ for $n \geq n_0$

$|a_{n+1} + a_{n+2} + \cdots + a_{n+m}| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+m}| < \varepsilon$

for $n \geq n_0$.

absolutely conv. [illegible] term [illegible] order [illegible] sum [illegible] [illegible] conditionally conv. [illegible]

[illegible]

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

19th [illegible] Dirichlet [illegible]

$$S' = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2m-1} - \frac{1}{4m-2} - \frac{1}{4m}\right)$$

$$S'_{3m} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{4m-2} - \frac{1}{4m}$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2m-1} - \frac{1}{2m}\right)$$

$$= \frac{1}{2} S_{2m}$$

$$\lim_{m\to\infty} S'_{3m} = \frac{1}{2}\lim_{m\to\infty} S_{2m} = \frac{1}{2}S$$

$$\lim S'_{3m+1} = \lim S'_{3m-1} = \lim S'_{3m}$$ [illegible] $\frac{1}{m+c}$ [illegible]

$$\therefore S' = \frac{1}{2}S.$$

[illegible] conditionally convergent series [illegible] no (infinity, oscillat. included) [illegible] (limit [illegible]) [illegible] Riemann [illegible]

$a_1 + a_2 + a_3 + \cdots$ [illegible] cond. conv. [illegible]

+ [illegible] term [illegible]

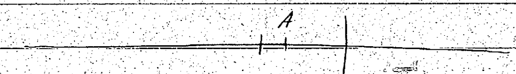
+ term $b_1 + b_2 + b_3 + \cdots$

term, $-c_1 - c_2 - c_3 - \cdots$

[illegible] diverge [illegible] conv. [illegible] $-c_1 - c_2 - c_3 - \cdots$ [illegible] $|c_1| + |c_2| + |c_3| + \cdots$ [illegible] conv. [illegible] series absolute conv. [illegible] div. [illegible] conv. [illegible] series, div. [illegible]

$a_1 + a_2 + a_3 + \cdots$ [illegible] conv. [illegible]

b_n [illegible] c_n [illegible] 0 [illegible] tend [illegible]



b term [illegible] A [illegible]

given no. [illegible] term [illegible] sum [illegible]

$b_1 + b_2 + \cdots + b_n$

[illegible] $-c_1 - c_2 - c_3 - c_4 - \cdots - c_n$ [illegible] A [illegible] term [illegible] c_{n+1} ~~[illegible]~~ [illegible]

[illegible] A [illegible] oscillat. [illegible] a_n [illegible] 0 [illegible] tend [illegible] b, div. [illegible] A [illegible] A [illegible] oscillat. [illegible]

Theorem. Absolutely conv. [illegible] term [illegible] limit [illegible]

[illegible] a [illegible] pack. [illegible]

$s = a_1 + a_2 + \cdots + a_n + \cdots$

$S_n = a_1 + a_2 + \cdots + a_n$

任意の順序 $s' = a'_1 + a'_2 + a'_3 + \cdots + a'_n + \cdots$

$S'_m = a'_1 + a'_2 + \cdots + a'_m$

S_n を [illegible] m を大きくとると S'_m 中に S_n の term が皆含まれる

$S'_m \geq S_n$

S'_m 中の term の a の suffix の最大を ν とし

$S_\nu \geq S'_m$

$S_\nu \geq S'_m \geq S_n \qquad \lim S_\nu = \lim S_n = s$

$S = S'$

次に一般に absolutely conv. s

$\sum a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ とする

$a_n = b_n - c_n, \qquad b_n, c_n \geq 0$

$$\begin{cases} a_n > 0, & c_n = 0 \\ a_n < 0, & b_n = 0 \end{cases}$$

と定義する

$\sum a_n = \sum (b_n - c_n)$

$\sum |a_n| = \sum (b_n + c_n)$

$\sum b_n$ conv.

$\sum c_n$ conv. [illegible]

故に順序を変えても S は一定

$\sum a'_n = \sum (b'_n - c'_n)$

$\sum |a'_n| = \sum (b'_n + c'_n)$

b'_n は b_n の、c'_n は c_n の [illegible]

$\sum b'_n = \sum b_n, \quad \sum c'_n = \sum c_n$

$\sum a'_n = \sum a_n$

Uniformly convergent series

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ ~~abs~~ conv.

の term が variable を含む一般

$u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots$

$= S(x) \qquad$ sum は $f(x)$ とする

之を differentiate 又は integrate する場合に each term の dif 又は int. したもの [illegible] infinite series

とする

$$\frac{d(\sin x)}{dx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

上は [illegible]

[illegible]

$$\int \sin x \, dx = C + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots = -\cos x$$

[illegible]

[illegible]

例ニ

$$U_n(x) = \frac{\sin nx}{n} - \frac{\sin (n-1)x}{n-1}$$

$$S_n(x) = \frac{\sin x}{1} + \left(\frac{\sin 2x}{2} - \frac{\sin x}{1}\right) + \cdots + \left(\frac{\sin nx}{n} - \frac{\sin (n-1)x}{n-1}\right)$$

$$= \frac{\sin nx}{n}$$

$$\lim S_n(x) = 0.$$

$$S(x) = 0.$$

$$U_n'(x) = \cos nx - \cos (n-1)x$$

$S_n'(x) = \cos nx$, $n \to \infty$, oscillating, not conv.

[illegible] Stokes, Seidel [illegible] physik [illegible] Uniform convergency [illegible] constant term ノ series [illegible] uniformly conv. [illegible] variable funct. [illegible] term ノ series [illegible] Integration ノ例ニ

$$u_n(x) = nxe^{-nx^2} - (n-1)\,xe^{-(n-1)x^2}$$

$$\int_0^1 u_n(x)\,dx = \frac{1}{2}\left(e^{-(n-1)x^2} - e^{-nx^2}\right)_0^1 = \frac{1}{2}\left(e^{-(n-1)} - e^{-n}\right)$$

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) = nxe^{-nx^2}$$

$$\lim S_n = 0$$

$u_1(x) + u_2(x) + \cdots$ 〃 $\quad S(x) = 0$

$$\int_0^1 S(x)\,dx = 0$$

$$\sigma(x) = \int_0^1 u_1(x)\,dx + \int_0^1 u_2(x)\,dx + \cdots + \int_0^1 u_n(x)\,dx + \cdots$$

$$\sigma_n(x) = \sum_1^n \frac{e^{-(n-1)} - e^{-n}}{2} = \frac{1 - e^{-n}}{2},$$

$$\lim_{n\to\infty} \sigma_n(x) = \sigma(x) = \frac{1}{2}.$$

Given $u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$

each term, one valued & continuous f. of x.

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

$$R_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots$$

$|f(x) - S_n(x)| < \varepsilon$ for $n \geqq n_0$

for all values of x $\quad a \leqq x \leqq b$

[illegible] infinite series ハ Uniformly convergent [illegible] n_0 ハ ε ノ funct. [illegible] x ニ independent [illegible]

例 $S(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ $\quad$ geometric series

$0 \leqq x < 1$ [illegible] conv.

$$S(x) = \frac{1}{1-x}$$

$$S_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

$$|S(x) - S_n(x)| = |R_n| = \left|\frac{x^n}{1-x}\right|$$

$0 \leqq x \leqq g < 1$ uniformly conv.

$$\frac{x^n}{1-x} < \varepsilon \quad \text{for } n \geqq n_0$$

$$x^n < \varepsilon(1-x)$$

~~$n \log$~~ $\quad \left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon(1-x)}$

$$n \log\left(\frac{1}{x}\right) > \log\frac{1}{\varepsilon} + \log\left(\frac{1}{1-x}\right)$$

$$n > \frac{\log\frac{1}{\varepsilon} + \log\frac{1}{1-x}}{\log\frac{1}{x}} < \frac{\log\frac{1}{\varepsilon} + \log\frac{1}{1-g}}{\log\frac{1}{g}} = n_0$$

[illegible]

$n \geq n_0$ ナル凡テノ場合ニ成リ立ツ

n_0 ハ ε ノ funct. デ x ニ independent, g ハ a constant ナリ

故ニ $0 \leq x \leq g$ デハ uniformly convergent ナリ

$0 \leq x < 1$ デハ g ヲ 1 ニ近ヅケレバ n_0 ハ

∞ ニ lim シ n_0 ハ x ニ depend ス

Theorem. Given $s_n = \sum u_n(x)$ ガ

$a \leq x \leq b$ uniform conv. ナルトキハ (suff. / without neces.)

$a \leq \alpha < \beta \leq b$ ナルトキ

$$\int_\alpha^\beta S(x)\,dx = \sum \int_\alpha^\beta u_n(x)\,dx$$

$$\int_\alpha^\beta S(x)\,dx = \int_\alpha^\beta S_n(x)\,dx + \int_\alpha^\beta R_n(x)\,dx$$

$$= \sum_{r=1}^{n} \int_\alpha^\beta u_r(x)\,dx + \int_\alpha^\beta R_n(x)\,dx$$

$$\left| \int_\alpha^\beta S(x)\,dx - \sum_{r=1}^{n} \int_\alpha^\beta u_r(x)\,dx \right| = \left| \int_\alpha^\beta R_n(x)\,dx \right| \leq \int_\alpha^\beta |R_n(x)|\,dx$$

$\because$ 一般ニ $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

$|R_n(x)| < \varepsilon$ for $n \geq n_0$

$$\int_\alpha^\beta |R_n(x)|\,dx < \varepsilon \int_\alpha^\beta dx = \varepsilon(\beta - \alpha) \quad \text{for } n \geq n_0$$

$$\left| \int_\alpha^\beta S(x)\,dx - \sum_{r=1}^{n} \int_\alpha^\beta u_r(x)\,dx \right| < \varepsilon(\beta - \alpha) \quad \text{for } n \geq n_0 \text{ indep. of } x$$

$\beta - \alpha$, finite.

$$\sum_{r=1}^{\infty} \int_\alpha^\beta u_r(x)\,dx = \int_\alpha^\beta S(x)\,dx$$

~~同様~~ 故ニ

$$S(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

$$\sigma(x) = u_1'(x) + u_2'(x) + \cdots + u_n'(x) + \cdots \text{ uniform conv. for } a \leq x \leq b$$

$$S'(x) = \sigma(x) = \sum u_n'(x)$$

$$\therefore \int \sigma(x)\,dx = S(x)$$

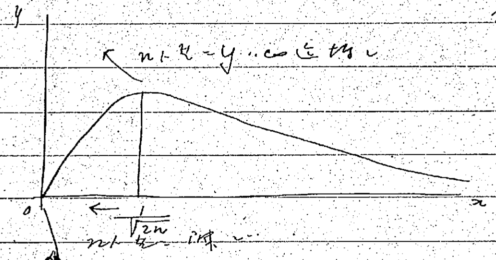
Ex. $u_n(x) = nxe^{-nx^2} - (n-1)xe^{-(n-1)x^2}$

$S_n(x) = nxe^{-nx^2}$

$S(x) = 0$

$R_n(x) = -nxe^{-nx^2}$

$y = S_n(x) = nxe^{-nx^2}$ $\quad y' = ne^{-nx^2} - 2n^2x^2e^{-nx^2} = 0 \quad x = \frac{1}{\sqrt{2n}}$

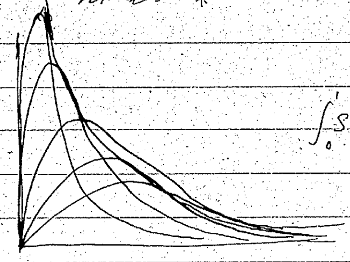


max. $x = \frac{1}{\sqrt{2n}}$

$y = \frac{\sqrt{n}}{\sqrt{2e}}$

$x \cdot y = \frac{1}{2\sqrt{e}}$

max. points locus



$n \to \infty$

$y = 0$ x-axis

$$\int_0^1 S(x)\,dx = 0 \neq \int_0^1 S_n(x)\,dx = \frac{1}{2}$$

Integration of power series

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ $\qquad u_n(x) = a_n x^n$

$|x| \leqq r$ convergent ⇒ term by term integration is admissible

$$\sum_{n=0}^{\infty} \int_{x_0}^{x} a_n x^n = \int_{x_0}^{x} f(x)\,dx$$

$$\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = \int f(x)\,dx$$

Proof.

$\sum a_n r^n$ converg. $|x| \leqq r$ $\quad \sum |a_n| r^n$

$|a_{n+1} r^{n+1} + a_{n+2} r^{n+2} + \cdots| < \varepsilon$ for $n \geqq n_0$

n_0, of course independent of x

$$|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots| \leqq |a_{n+1}| r^{n+1} + |a_{n+2}| r^{n+2} + \cdots$$

$< \varepsilon$ for $n \geqq n_0$

original series, uniformly convergent

hence the theorem

定理.. power series が収束する

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x| < 1$$

$$|x| < \lim\left|\frac{a_n}{a_{n+1}}\right|$$

$r < \lim\left|\frac{a_n}{a_{n+1}}\right|$ ならば収束する

$|x| \leqq r$ ⇒ $\sum |a_n||x^n|$ absolutely convergent.

Ex. Elliptic integral of the 1st kind.

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad (0<k^2<1)\ (0 \leqq x \leqq 1)$$

$x = \sin\varphi$

$$F(k,\varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

φ: amplitude

k: modulus of the ellip. int.

$(1-k^2\sin^2\varphi)^{-\frac12}$ expand by binomial theorem

$$= 1 + \frac12 k^2\sin^2\varphi + \frac{1\cdot3}{2\cdot4}k^4\sin^4\varphi + \cdots + \frac{1\cdot3\cdot5\cdots(2h-1)}{2\cdot4\cdot6\cdots2h} k^{2h}\sin^{2h}\varphi + \cdots$$

convergent, integrable term by term.

$$F(k,\varphi) = \varphi + \frac12 k^2\int_0^\varphi \sin^2\varphi\,d\varphi + \frac{1\cdot3}{2\cdot4}k^4\int_0^\varphi\sin^4\varphi\,d\varphi + \cdots + \frac{1\cdot3\cdot5\cdots 2h-1}{2\cdot4\cdot6\cdots2h}k^{2h}\int_0^\varphi \sin^{2h}\varphi\,d\varphi + \cdots$$

k: sufficiently small ⇒ ($k=0$, $\arcsin\varphi$)

$\varphi + \frac12 k^2\int_0^\varphi \sin^2\varphi\,d\varphi$,

$$F\left(k,\frac\pi2\right) = F(k) = \int_0^{\frac\pi2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

this important constant.

$$\int_0^{\frac\pi2}\sin^{2h}\varphi\,d\varphi = \left(-\sin^{2h-1}\varphi\cos\varphi\right)_0^{\frac\pi2} + (2h-1)\int_0^{\frac\pi2}\sin^{2h-2}\varphi\cos^2\varphi\,d\varphi$$

$$\cancel{2h}\int_0^{\frac\pi2}\sin^{2h}\varphi\,d\varphi = \frac{(2h-1)}{2h}\int_0^{\frac\pi2}\sin^{2h-2}\varphi\,d\varphi$$

$$\int_0^{\frac\pi2}\sin^{2h}\varphi\,d\varphi = \frac{2h-1}{2h}\cdot\frac{2h-3}{2h-2}\cdot\frac{2h-5}{2h-4}\cdots\frac34\cdot\frac12\int_0^{\frac\pi2}d\varphi$$

$$\therefore F(k) = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdot 6\cdots 2n}\right)^2 k^{2n} + \cdots\right\}$$

[illegible] $F(k)$ [illegible]

k [illegible] 1 [illegible]

k^2 [illegible] $1-k^2 = k'^2$ [illegible] x [illegible] change of variable [illegible]

Elliptic integral of the 2nd kind

$$\int_0^x \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx = \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\, d\varphi = E(k,\varphi)$$

$$(1-k^2\sin^2\varphi)^{\frac{1}{2}} = 1-\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots 2n-3}{2\cdot 4\cdot 6\cdots 2n-2}\frac{1}{2n}k^{2n}\sin^{2n}\varphi$$

$$\int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\,d\varphi = \varphi - \frac{1}{2}k^2\int_0^\varphi \sin^2\varphi\, d\varphi - \cdots - \frac{1\cdot 3\cdot 5\cdots 2n-3}{2\cdot 4\cdot 6\cdots 2n-2}\frac{1}{2n} \times k^{2n}\int_0^\varphi \sin^{2n}\varphi\, d\varphi - \cdots$$

$$E(k) = \int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,d\varphi = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx = \frac{\pi}{2}\left\{1-\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdot 6\cdots 2n}\right)^2 \times k^{2n}\right\}$$

$F(k,\varphi)$, $F(k)$, $E(k,\varphi)$, $E(k)$ [illegible] Jahnke-Emde's Funktionentafeln [illegible] Legendre [illegible]

Extension of the notion of definite integrals

def [illegible] introduce [illegible]

$\int_a^b f(x)\,dx$, 1. a, b finite

2. $f(x)$, one-valued continuous in $a \leqq x \leqq b$.

[illegible]

1. infinite interval, 2. interval [illegible] limit [illegible] integrand [illegible] discontinuous 3. interval [illegible] discontinuity [illegible]

1. Infinite interval

$f(x)$ [illegible] $x_0 \geqq a$, (a finite) [illegible] one val. cont. [illegible]

$\int_a^{x_0} f(x)\,dx$ [illegible] definite meaning [illegible] x [illegible] variable [illegible] $x \geqq a$ [illegible]

$\lim_{x\to+\infty}\int_a^x f(x)\,dx$ [illegible] exist [illegible]

$\int_a^\infty f(x)\,dx$ [illegible] integral [illegible] x, ∞ [illegible] convergent [illegible]

[illegible]

$f(x)$ [illegible] $x \leqq b$ [illegible] one val. cont. [illegible]

$\lim_{x\to-\infty}\int_x^b f(x)\,dx$ [illegible] $= \int_{-\infty}^b f(x)\,dx$ [illegible]

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{x\to+\infty}\int_0^x \frac{dx}{1+x^2} = \lim_{x\to\infty}\{\arctan x - \arctan 0\} = \frac{\pi}{2}$$

$$\int_0^\infty e^{-x}x^{n-1}dx = \lim_{x\to\infty}\int_0^x e^{-x}x^{n-1}dx$$

$$\int_0^x e^{-x}x^{n-1}dx = \left(-e^{-x}x^{n-1}\right)_0^x + (n-1)\int_0^x e^{-x}x^{n-2}dx$$
$$= -e^{-x}x^{n-1} - (n-1)e^{-x}x^{n-2} + (n-1)(n-2)\int_0^x e^{-x}x^{n-3}dx$$
$$= -e^{-x}\left\{x^{n-1} + (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} + \cdots \right.$$
$$\left. + (n-1)(n-2)\cdots 3\cdot 2\cdot 1\int_0^x e^{-x}dx\right\}$$
$$= -e^{-x}\left\{x^{n-1} + (n-1)x^{n-2} + \cdots + (n-1)\cdots 3\cdot 2\cdot 1\right\} + (n-1)!$$

higher / lower

$$\lim_{x\to+\infty} -e^{-x}\left\{x^{n-1} + \cdots\right\} = 0$$

$$\int_0^\infty e^{-x}x^{n-1}dx = n!$$

$\lim_{l\to+\infty}\int_a^l f(x)dx$ [illegible] integral [illegible]

[illegible] criterion [illegible]

[illegible]

$$\int_a^l \frac{dx}{x^\mu}\ (a>0),\quad f(x) = \frac{1}{x^\mu},\quad \text{one val. cont.}$$

$$= \frac{1}{1-\mu}\left(\frac{1}{l^{\mu-1}} - \frac{1}{a^{\mu-1}}\right),\quad \mu \neq 1$$

$\mu > 1$: $\lim_{l\to+\infty}\int_a^l \frac{dx}{x^\mu} = \frac{1}{\mu-1}\cdot\frac{1}{a^{\mu-1}} = \int_a^{+\infty}\frac{dx}{x^\mu}$

$\mu < 1$: $\lim_{l\to+\infty} \to \infty$, $\mu = 1$, $\log x$, [illegible] $x\to$, ∞

$\int_a^{+\infty}\frac{dx}{x^\mu}$ [illegible] $\mu > 1$ [illegible]

generally

$\int_a^\infty f(x)dx$, $f(x) = \frac{\varphi(x)}{x^\mu}$ $\varphi(x)$ finite for $x\to+\infty$

then $\int_a^\infty f(x)dx$ exists when $\mu > 1$

$$\int_a^{l'} f(x)dx - \int_a^l f(x)dx = \int_l^{l'} f(x)dx \qquad l' > l$$

$|\varphi(x)| \leq K$ $(a, +\infty)$ [illegible] K exist

$$\left|\int_l^{l'} f(x)dx\right| = \left|\int_l^{l'}\frac{\varphi(x)}{x^\mu}dx\right| \leq K\left|\int_l^{l'}\frac{dx}{x^\mu}\right| \to 0 \text{ for } l\to+\infty$$

$$K\left|\int_l^{l'}\frac{dx}{x^\mu}\right| < \varepsilon \quad \text{for } l \geqq g$$

$\therefore \int_a^{+\infty} f(x)dx$ exists when $\mu > 1$.

sufficient but not necessary cond.

$x^\mu f(x) = \varphi(x)$, finite $(a, +\infty)$ ~~[illegible]~~ [illegible]

μ [illegible] > 1 [illegible]

Ex.

$\int_0^\infty \frac{\cos ax}{1+x^2}dx$ [illegible] indefinite int. [illegible]

$$f(x) = \frac{\cos ax}{1+x^2}$$

$x^2 f(x) = \frac{x^2\cos ax}{1+x^2} = \varphi(x) = \frac{\cos ax}{1+\frac{1}{x^2}}$, $\cos ax$ oscillate but remains finite when $x\to\infty$

$\varphi(x)$ finite.

$\mu = 2 > 1$, $\therefore \int_a^\infty f(x)dx$ exist.

$f(x)$ continuous, one valued for $a \leqq x < b$

$f(x)$ discont. for $x = b$

$\int_a^b f(x)\,dx$, whether it has sense or not?

for ex. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, discont. at $x=1$.

$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ "

$\int_a^{b-\varepsilon} f(x)\,dx$, exists indeed

When $\lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)\,dx$ exists;

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)\,dx$$

Similarly

$f(x)$ cont. $a < x \leq b$.

discont. $x = a$

$\lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)\,dx = \int_a^b f(x)\,dx$, when the limit exists.

Ex. $\int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} = (\arcsin x)_0^{1-\varepsilon} = \arcsin(1-\varepsilon) \longrightarrow \frac{\pi}{2}$ for $\varepsilon\to 0$

$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

Criterion for the existence of the limit.

$\int_a^b f(x)\,dx$, $f(x)$ disc. at $x=b$.

$$\int_a^{b-\varepsilon}\frac{dx}{(x-b)^\mu} = \left(\frac{1}{1-\mu}\frac{1}{(x-b)^{\mu-1}}\right)_a^{b-\varepsilon}$$

$$= \frac{1}{1-\mu}\left\{\frac{1}{(-\varepsilon)^{\mu-1}} - \frac{1}{(a-b)^{\mu-1}}\right\} \qquad \mu \neq 1.$$

$\mu < 1.$ $\frac{1}{(-\varepsilon)^{\mu-1}} \to 0.$

$$\int_a^b \frac{dx}{(x-b)^\mu} = -\frac{1}{1-\mu}\frac{1}{(a-b)^{\mu-1}} \quad \text{exists.}$$

$\mu > 1.$ $\frac{1}{(-\varepsilon)^{\mu-1}} \to \infty.$ $\mu = 1.$ $\log \varepsilon$. $\log_{-\varepsilon} \infty.$

generally

sufficient cond. for the exist. of $\int_a^b f(x)\,dx$

$f(x) = \frac{\varphi(x)}{(x-b)^\mu}$, $\varphi(x)$ remains finite in $a \leq x \leq b$.

$+\ \mu < 1.$

$f(x)$, disc. at $x = a$.

$f(x) = \frac{\varphi(x)}{(x-a)^\mu}$, — — — — — — — — .

$(x-b)^\mu f(x)$ finite at $x = b$. + some μ, $\mu < 1$

Ex. $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

$$f(x) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} = \underbrace{\frac{1}{\sqrt{(1+x)(1-k^2x^2)}}}_{\text{finite } x=1}\cdot\frac{1}{\sqrt{1-x}}$$

$(1-x)^{\frac{1}{2}}$ infinity of order of $\frac{1}{2}$.

$$f(x)(1-x)^{\frac{1}{2}} = \frac{1}{\sqrt{(1+x)(1-k^2x^2)}} = \varphi(x),\ \text{finite} \quad \mu < 1.$$

$$\int_0^{+\infty} x^{p-1}e^{-x}\,dx, \quad (p>0), = \Gamma(\alpha) \quad \text{Gamma funct.}$$

此ハ definite integral ニテ define セラレタ funct. ニシテ [illegible] 此ハ exist スルヤ [illegible]

$$f(x) = x^{p-1}e^{-x}$$
$$x^{\mu}f(x) = x^{\mu+p-1}e^{-x}$$

x 大ナルトキ e^{-x} ハ $-\infty$ ノ order 高キ故 $x^{\mu}f(x)$ 0 ニ tend シ

$\mu > 1$ ニテ [illegible] finite ナル故

$$\int_a^{+\infty} f(x)\,dx \quad \text{exist}$$

x^{p-1} ハ $1 > p > 0$ ニテ discont. at $x=0$ ナル故

[illegible]

$$x^{1-p}f(x) = e^{-x}$$

此ハ $x=0$ ノ近傍ニテ finite ナリ

x^{1-p} $1-p < \mu$ ナル故 $\int_0^b f(x)\,dx$ exist

故ニ一般ニ $\int_0^{+\infty} x^{1-p}e^{-x}\,dx$ exist for $p > 0$.

此ノ [illegible] integ. by parts ヲ →

$$\int_0^{+\infty} e^{-x}x^{1-p}\,dx = \left(-e^{-x}x^{p-1}\right)_0^{\infty} + (p-1)\int_0^{\infty} e^{-x}x^{p-2}\,dx \qquad p > 1$$

$$\Gamma(p) = (p-1)\,\Gamma(p-1) \qquad \text{fundamental property}$$

此 funct. ハ functional equat. ニテ define セラレタル [illegible]

If p positive integer

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2) = (n-1)!\,\Gamma(1)$$
$$= (n-1)!$$

Gamma f. ハ factorial ヲ一般ニ extend シタルモノ [illegible] exponential f. ハ ellipt. f. hyperellipt. f. ハ differential eq. ニテ define サル。此 Gamma f. ハ dif. eq. ヲ satisfy セザル [illegible]

$y = \Gamma(x)$

y' $\qquad G(x, y, y', \ldots y^{(n)} \ldots) = 0$

y'' $\qquad$ ナル rational ff f. ノ eq. [illegible]

$\vdots$ $\qquad$ ([illegible] integral eq. ヲ

$y^{(n)}$ $\qquad$ polynomial) →

[illegible] transcendentally transcendental f. (Hypertransc. f.) [illegible]

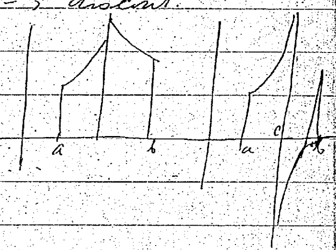
algebraically transc. f. [illegible]

funct. eq. ニテ define セラレタ f. ノ研究ハ [illegible] 1900 年 integral eq. ト dif. eq. [illegible] Integro-diff. eq. [illegible]

3 $f(x)$ $\quad a \leq x \leq b$

$a < c < b$ [illegible] c [illegible] discont.

$$\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon'}^{b} f(x)\,dx, \text{ definite}$$



$\varepsilon \rightarrow 0$

$\varepsilon' \rightarrow 0$ [illegible] limit [illegible]

$$\int_a^b f(x)\,dx \text{ [illegible]}$$

[illegible] ε [illegible] ε' [illegible] independently tend toward 0.

[illegible] case [illegible] apply [illegible]

Criterion,

$(x-c)^{\mu} f(x)$ $\quad \mu < 1$, $x = c$ [illegible] finite [illegible]

limit [illegible]

[illegible] case 2 [illegible]

[illegible] extend [illegible] a [illegible] b [illegible] finite no. of discontinuous pts [illegible]

Ex $\int_{-1}^{+1} \frac{dx}{x}$ .. [illegible]

$$\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{+\varepsilon'}^{+1} \frac{dx}{x},$$

$$= \int_1^{\varepsilon} \frac{dx}{x} + \int_{\varepsilon'}^{1} \frac{dx}{x} = \log \varepsilon - \log \varepsilon'$$

$\left.\begin{array}{l}\varepsilon \rightarrow 0 \\ \varepsilon' \rightarrow 0\end{array}\right\}$ $\log \frac{\varepsilon}{\varepsilon'}$ [illegible] limit [illegible], 0 [illegible] order [illegible] limit [illegible]

[illegible] dif. [illegible] $\int_{-1}^{+1} \frac{dx}{x}$ .. does not exist,

[illegible] $\varepsilon = \varepsilon' \rightarrow 0$ [illegible] $\log \varepsilon - \log \varepsilon' = 0$ [illegible]

$\int_{-1}^{+1} \frac{dx}{x} = 0$, [illegible] principal value of $\int_{-1}^{+1} \frac{dx}{x}$ [illegible]

(by Cauchy)

[illegible]

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{g \rightarrow \infty \\ g' \rightarrow \infty}} \int_{-g'}^{g} f(x)\,dx$$ [illegible]

$g = g' \rightarrow \infty$ [illegible] exist [illegible]

[illegible] limit [illegible] principal value of the integral [illegible]

Quadrature & rectification of curves

$\int_a^b f(x)\,dx$ $\quad y = f(x)$ [illegible] area [illegible]

$f(x)$ one valued [illegible] x [illegible] one value

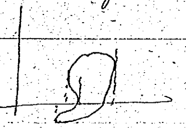
[illegible] y [illegible] one value [illegible]

Curve [illegible] x axis [illegible] x axis [illegible]

[illegible] positive [illegible] negative [illegible] algebraic sum [illegible]

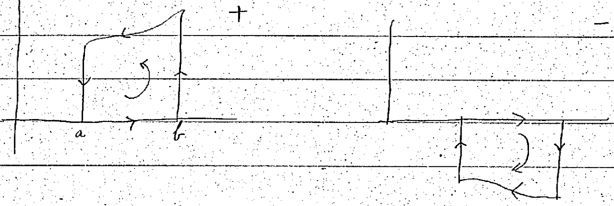
Curve [illegible] one valued f. [illegible] closed curve [illegible] quadrature [illegible]

[illegible] one valued f. [illegible] tangent [illegible] y axis [illegible] parallel [illegible]



curve tracing

[illegible] eq. [illegible] axis [illegible]



integral $= +$, $-$ [illegible] area $= +, -$ [illegible]

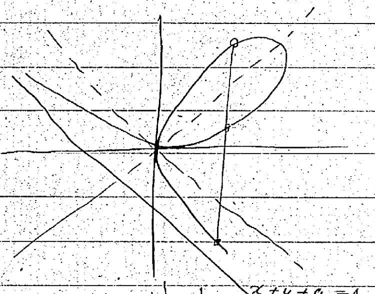
[illegible] $\int_a^b y\,dx$: area [illegible] algebraic [illegible]

Descartes' folium

$$x^3+y^3-3axy=0$$

$$r=\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$$

$$x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3}\qquad t \text{ parameter}$$

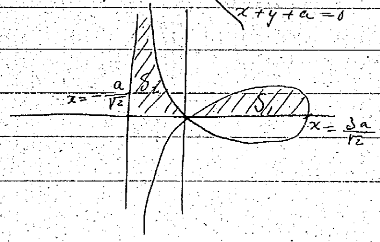


x [illegible] one val. [illegible] y [illegible] 3 values

[illegible] loop [illegible] area [illegible] branch

[illegible] asis

[illegible] (45°)



$$x \to \frac{x+y}{\sqrt2}$$

$$y \to \frac{x-y}{\sqrt2}$$

$$\left(\frac{x+y}{\sqrt2}\right)^3+\left(\frac{x-y}{\sqrt2}\right)^3-3a\,\frac{x^2-y^2}{2}=0$$

$$y^2=\frac{x^2(3a-\sqrt2\,x)}{3(a+\sqrt2\,x)}$$

$$S_1=\int_0^{\frac{3a}{\sqrt2}} y\,dx \qquad S_2=\int_{-\frac{a}{\sqrt2}}^0 y\,dx$$

$$=\int_0^{\frac{3a}{\sqrt2}}\frac{x}{\sqrt3}\sqrt{\frac{3a-\sqrt2 x}{a+\sqrt2 x}}\,dx \qquad S_2=\int_{-\frac{a}{\sqrt2}}^0 -\frac{x}{\sqrt3}\sqrt{}$$

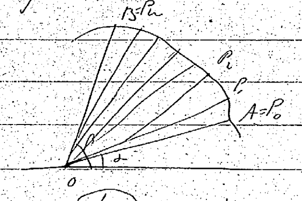
$\sqrt2\,x=at$ $\qquad x=-\frac{a}{\sqrt2}$ [illegible] infinite [illegible]

$$\sqrt{\frac{3a-\sqrt2 x}{a+\sqrt2 x}}=\sqrt{\frac{3-t}{1+t}}\qquad u=\tan\theta$$

$$S_1=\frac{4a^2}{\sqrt3}\int_0^{\sqrt3}\frac{(3-u^2)u^2\,du}{(1+u^2)^3}=\frac{4a^2}{\sqrt3}\int_0^{\frac{\pi}{3}}\ldots 3\theta\,d\theta=\frac{3a^2}{4}$$

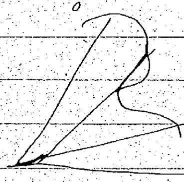
$$S_2=\frac{3a^2}{4}$$

by Polar Coordinates



$r=f(\theta)$ one valued +

Cont. at (α,β)



[illegible] many valued [illegible]

$\angle x_0 O P_k=\theta_k \qquad \theta_0=\alpha,\ \theta_n=\beta$

$\theta_k > \eta_k > \theta_{k-1}$ [illegible] any angle

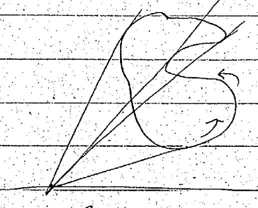
η_k [illegible]

[illegible] radius vector [illegible] circular arc [illegible]

[illegible] sum [illegible] n [illegible]

$$\lim\sum\frac12 f(\eta_k)^2(\theta_k-\theta_{k-1})$$

$$=\int_\alpha^\beta \frac12 r^2\,d\theta$$



tangent Rad. vector

によって area を正とする Curve

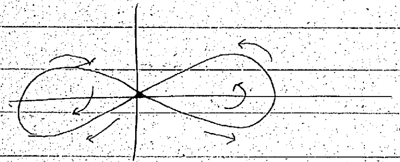
の回る方向 positive direct.

とる

Lemniscate

$$(x^2+y^2)^2 = 2a^2(x^2-y^2)$$

$$r^2 = 2a^2\cos 2\theta$$



$$\int_0^{2\pi}\frac{1}{2}r^2\,d\theta = 0$$

$$\int_0^{\frac{\pi}{4}}\frac{1}{2}r^2\,d\theta$$

Parametric representat.

$x = \varphi(t)$ $\qquad \varphi,\ \psi$ interval cont.

$y = \psi(t)$ $\qquad (t_0,\ t_1)$

polar coord. $r^2 = x^2+y^2 = \varphi^2+\psi^2$

$$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \qquad \frac{y}{x} = \tan\theta = \frac{\psi(t)}{\varphi(t)}$$

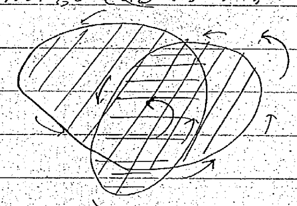
$$\sec^2\theta\,d\theta = \frac{\varphi(t)\psi'(t)-\varphi'(t)\psi(t)}{\varphi^2(t)}\,dt$$

$$\frac{\varphi^2+\psi^2}{\varphi^2}$$

$$d\theta = \frac{\varphi(t)\psi'-\varphi'\psi}{\varphi^2+\psi^2}\,dt$$

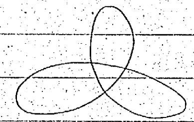
$$\int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_{t_0}^{t_1}(\varphi\psi'-\varphi'\psi)\,dt = \frac{1}{2}\int(xy'-x'y)\,dt$$

此ノ場合ニハ面積ノ方向ヲ考ヘテ居ル

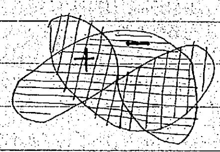


$x = r(2\cos 2\theta - \cos\theta)$

$y = r(2\sin 2\theta + \sin\theta)$



$\int_0^{2\pi}(xy'-x'y)\,dt$. Sum トス



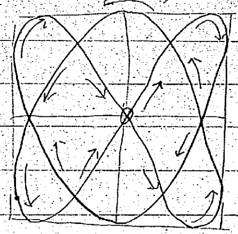
Combination of simple harmonic motion.

$x = r\sin 2\theta$ $\qquad$ Lissajous curves

$y = r\sin 3\theta$ $\qquad$ 一般ニ $x = a\sin(mt+\alpha)$

$y = b\sin(nt+\beta)$

m, n ガ rational ratio ナレバ closed curve トナル



$$\frac{1}{2}\int_0^{2\pi}(xy'-x'y)\,dt = \frac{a^2}{2}\int_0^{2\pi}\{3\sin 2t\cos 3t - 2\sin 3t\cos 2t\}\,dt = 0$$

一般ニ generally $\int_0^{2\pi}\sin ht\cos kt\,dt = 0$ $\quad h\neq k$

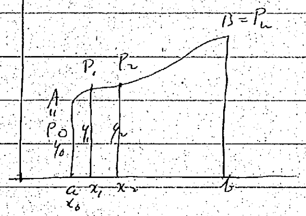
故ニ両方ノ面積ガ打チ消ス

Rectification

curve ト equal length ノ straight line ヲ見出スコトヲ云フ

$y = f(x)$ (a,b) interval cont.

& $f'(x)$ also (a,b) interval cont.



[illegible] curve length [illegible] define [illegible] curve [illegible] n pts [illegible] inscribed polygon [illegible] (broken line) [illegible] n [illegible] point [illegible] dist 0 (tend) [illegible] limit [illegible] curve [illegible]

$$P_0P_1 + P_1P_2 + \cdots + P_{n-1}P_n = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$\sum_{k=1}^{n} (x_k - x_{k-1}) \sqrt{1 + \left(\frac{y_k - y_{k-1}}{x_k - x_{k-1}}\right)^2} \qquad y = f(x)$$

$$\frac{y_k - y_{k-1}}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y}{\Delta x}$$

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_{k-1} + \theta(x_k - x_{k-1})) \quad \text{by mean value theorem}$$

funct.

$$= f'(x_{k-1}) + \varepsilon_k(x_k - x_{k-1}) \qquad \text{for } f' \text{ is cont.}$$

$$\varepsilon_k \to 0 \text{ when } (x_k - x_{k-1}) \to 0$$

$$\Sigma = \sum (x_k - x_{k-1}) \sqrt{1 + (f'(x_{k-1}) + \varepsilon_k)^2}$$

$$= \sum (x_k - x_{k-1}) \left\{ \sqrt{1 + f'(x_{k-1})^2} + \delta \right\} \qquad \delta(b-a)$$

$$= \sum (x_k - x_{k-1}) \sqrt{1 + f'(x_{k-1})^2} + \delta \sum (x_k - x_{k-1}) \qquad \delta \to 0$$

$$\lim \Sigma = \int_a^b \sqrt{1 + f'(x)^2}\, dx \quad \text{limit really exists + has this value}$$

$$S = \int_a^b \sqrt{1 + y'^2}\, dx$$

$$S = \int_a^x \sqrt{}\, dx$$

$$\frac{ds}{dx} = \sqrt{1 + y'^2}$$

if $x = \varphi(t) \qquad y = \psi(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'}{\varphi'}$$

$$S = \int_{t_0}^{t_1} \sqrt{1 + \frac{\psi'^2}{\varphi'^2}}\, \varphi'\, dt$$

$$= \int_{t_0}^{t_1} \sqrt{\varphi'^2 + \psi'^2}\, dt$$

if $x = r\cos\theta$

$y = r\sin\theta \qquad r = f(\theta)$

$$\varphi' = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\psi' = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\varphi'^2 + \psi'^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$S = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + r'^2}\, d\theta$$

Space curve [illegible]

$$\left.\begin{aligned} y &= f(x) \\ z &= \varphi(x) \end{aligned}\right\} \qquad S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

$x = f(t)$

$y = \varphi(t) \qquad S = \int \sqrt{f'^2 + \varphi'^2 + \psi'^2}\, dt$

$z = \psi(t)$

Ex. 1. Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$x = a \operatorname{cosec} \theta$

$y = b \cot \theta$

$$S = \int \sqrt{a^2 \operatorname{cosec}^2\theta \cot^2\theta + b^2 \operatorname{cosec}^4\theta}\, d\theta$$

$$= \int \frac{\sqrt{a^2\cos^2\theta + b^2}}{\sin^2\theta}\, d\theta = \int \frac{\sqrt{a^2+b^2-a^2\sin^2\theta}}{\sin^2\theta}\, d\theta$$

$$= \sqrt{a^2+b^2}\int \frac{\sqrt{1-\frac{a^2}{a^2+b^2}\sin^2\theta}}{\sin^2\theta}\, d\theta \qquad \frac{a^2}{a^2+b^2} = k^2$$

$$\int \frac{\sqrt{1-k^2\sin^2\theta}}{\sin^2\theta}\, d\theta = -\cot\theta\sqrt{1-k^2\sin^2\theta} + k^2\int \frac{\cot\theta \sin\theta\cos\theta}{\sqrt{}}\, d\theta \qquad \begin{aligned} &k^2 - k^2\sin^2\theta \\ &= -1(1-k^2\sin^2\theta) + k^2 - 1 \end{aligned}$$

$$= -\cot\theta\sqrt{1-k^2\sin^2\theta} + \underbrace{\int\sqrt{1-k^2\sin^2\theta}\, d\theta}_{\text{Ell. I. 2nd k.}} + (k^2-1)\underbrace{\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}}_{\text{Ellip. int. 1 kind}}$$

2. Intersection curve of a circular cylinder $x^2+y^2=a^2$ & a hyperbolic paraboloid $z = 2xy$

$$\begin{cases} x^2+y^2 = a^2 \\ z = 2xy \end{cases} \qquad \begin{aligned} &x = a\cos\theta \quad y = a\sin\theta \\ &z = 2a^2\sin\theta\cos\theta = a^2\sin 2\theta \end{aligned}$$

$$S = \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi}\sqrt{a^2+4a^4\cos^2 2\theta}\, d\theta$$

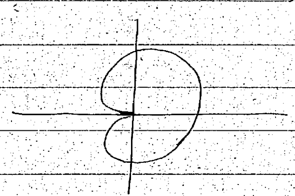
$$= \int_0^{2\pi}\sqrt{a^2+4a^4-4a^4\sin^2 2\theta}\, d\theta$$

$$= \frac{\sqrt{a^2+4a^4}}{2}\int_0^{4\pi}\sqrt{1-\frac{4a^4}{4a^4+a^2}\sin^2\varphi}\, d\varphi \qquad \varphi = 2\theta$$

Elliptic integral of the 2nd Kind

Remark $\quad s = \int_a^b \left|\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right| dx \quad \because$ positive [illegible]

$r = a(1+\cos\theta)$ $\qquad$ Cardioid



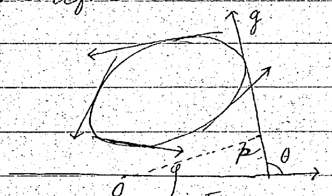
$$S = \int_0^{2\pi}\sqrt{r^2+r'^2}\, d\theta$$

$$= a\int_0^{2\pi}\sqrt{2+2\cos\theta}\, d\theta$$

$$= a\int_0^{2\pi}\left|2\cos\frac{\theta}{2}\right| d\theta$$

$0 - \pi$, $+$ $\qquad$ $\pi - 2\pi$, $-$

Rectification [illegible] equation [illegible]

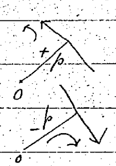


θ, p [illegible] tangent [illegible]

$p = f(\theta)$ [illegible] curve defin[illegible]

tangent [illegible] envelop [illegible]

[illegible] line coordinates



Tangent eq. $\begin{cases} x\sin\theta - y\cos\theta = p \quad [x\cos\varphi + y\sin\varphi = p] \\ p = f(\theta) \end{cases}$

envelop $\begin{cases} x\sin\theta - y\cos\theta = p(\theta) \\ x\cos\theta + y\sin\theta = p'(\theta) \end{cases}$

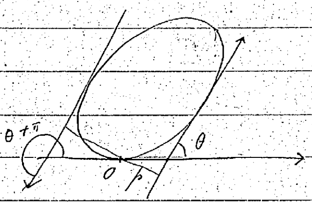
$$\begin{cases} x = p(\theta)\sin\theta + p'(\theta)\cos\theta \\ y = -p(\theta)\cos\theta + p'(\theta)\sin\theta \end{cases}$$

$$\frac{dx}{d\theta} = p(\theta)\cos\theta + p''(\theta)\cos\theta = \cos\theta\,(p+p'')$$

$$\frac{dy}{d\theta} = p(\theta)\sin\theta + p''(\theta)\sin\theta = \sin(\theta)(p+p'')$$

$$S = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int |(p+p'')|\, d\theta$$

closed + convex curve



$p(\theta)$ 十 $p(\theta+\pi)$ ノ対 和ノ中 此ガ一定ナル如キ Curve ヲ Curve of constant breadth ト云フ 此ニ

就イテ研究セラレシ一寸

$$p(\theta) + p(\theta+\pi) = k$$

Circle ガ其ノ一例ニシテ curve ハ square ノ中ニ inscribe シテ常ニ touch シ、回転シ得ル Reuleaux ノ Kinematics 中ニ記載サレ Mechanik ニ応用サル

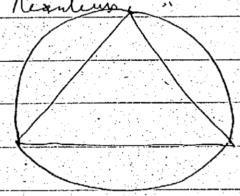
$$p(\theta) = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) + (a_3\cos 3\theta + b_3\sin 3\theta) + \cdots + (a_{2m+1}\cos(2m+1)\theta + b_{2m+1}\sin(2m+1)\theta) + \cdots$$

$$p(\theta+\pi) = \frac{a_0}{2} - (\qquad) - (\qquad) - (\qquad) - \cdots$$

$$p(\theta) + p(\theta+\pi) = a_0$$

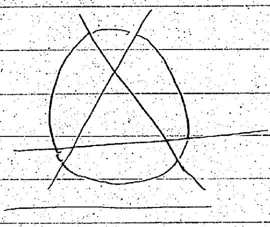
無数ニ作レル

又 Reuleaux ノ 正三角形



此ハ Curve of const. breadth. vertex ヲ Centre トシテ対辺ノ両端ヲ通ル circular arc. 実用ニ於テ simple ナリ

或ハ



(tangent continuous ナリ)

$p(\theta) + p(\theta+\pi) = k$ ヲ満ス Curve of constant breadth ノ周囲ハ一定デアル

$$S = \int_0^{2\pi} (p+p'')\, d\theta \quad \text{circle ト同ジナリ}$$

$$= \int_0^{2\pi} p(\theta)\, d\theta + \underbrace{(p')_0^{2\pi}}_{0}$$

$$= \int_0^{\pi} p(\theta)\, d\theta + \int_{\pi}^{2\pi} p(\theta)\, d\theta = \int_0^{\pi} p(\theta)\, d\theta + \int_0^{\pi} p(\theta+\pi)\, d\theta$$

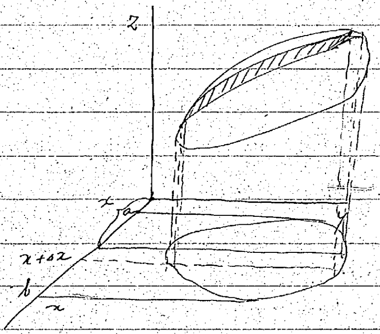
$$= \int_0^{\pi} (p(\theta) + p(\theta+\pi))\, d\theta = k\pi \quad \text{ハ constant}$$

此ヲ line coord ニヨリテ求メルニハ

Volume & area of a surface

一般に double integral [illegible] single integral [illegible]

[illegible]



[illegible] cylinder の vol [illegible]

$A(x)$, yz // [illegible] x [illegible] plane

の section の area

$A(x)\Delta x$ [illegible] segment の vol

$\sum A(x)\Delta x$

$\lim \sum A(x)\Delta x = V$

$\int_a^b A(x)\,dx = V$

Ex. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$ ellipse

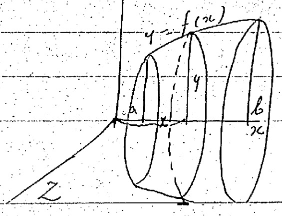
$\frac{y^2}{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2} + \frac{z^2}{\left(c\sqrt{1-\frac{x^2}{a^2}}\right)^2} = 1$

$Ax = \pi bc\left(1 - \frac{x^2}{a^2}\right)$

$V = \int_{-a}^{+a} \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \pi bc\left\{x - \frac{x^3}{3a^2}\right\}_{-a}^{+a} = \frac{4\pi abc}{3}$

一般に surface of revolution [illegible] principle [illegible]

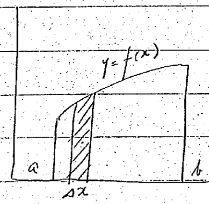
[illegible]



$y = f(x)$ [illegible] x axis [illegible] revolution [illegible]

$A(x) = \pi y^2$

$V = \pi \int_a^b y^2\,dx$



centre of grav (ξ, η)

$\frac{y}{2}\ \underset{\text{mass}}{y\Delta x}$

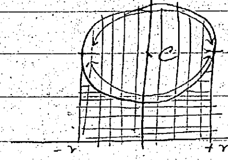
$\eta = \frac{\sum \frac{y}{2} y\Delta x}{\sum y\Delta x} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$

$V = 2\pi\eta \int_a^b y\,dx$ ###

$V = p \cdot A$ $\qquad 2\pi\eta$ perimeter of a circle of η.

これ Guldin's rule という

Ex. Torus (circular ring)



$x^2 + (y-c)^2 = r^2$

$(c > r)$

$y = c \pm \sqrt{r^2 - x^2}$

$y_1 = c + \sqrt{r^2 - x^2}$

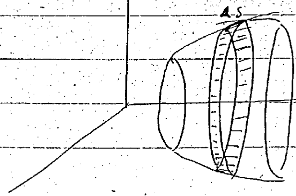
$y_2 = c - \sqrt{r^2 - x^2}$

$V = \int_{-r}^{+r} \pi(y_1^2 - y_2^2)\,dx = 4c\pi \int_{-r}^{+r} \sqrt{r^2 - x^2}\,dx = \cancel{4}^{2}\pi c \cdot \frac{\pi r^2}{2}$

or by Guldin's rule

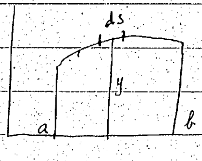
centre of gr. c. $\quad V = 2\pi c \cdot \pi r^2$

area of surface of revol.



$A = \lim \sum 2\pi y\Delta s$

$= \int_a^b 2\pi y\,ds$



centre of grav. of the curve (ξ, η)

$$\eta = \frac{\Sigma y\,ds}{\Sigma ds} = \frac{\int_a^b y\,ds}{\int_a^b ds}$$

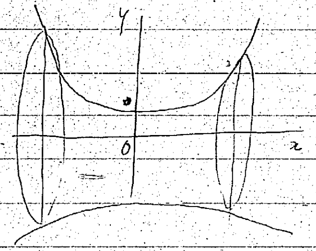
$A = 2\pi\eta\int_c^b ds = p \cdot s$ s. curve length.

this is a Guldin's rule.

Torus: c curve, c. of gr.

$$A = 2\pi c \cdot 2\pi r = 4\pi^2 c r.$$

Ex. Catenary $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$



$$A = 2\pi\int_0^x y\,ds \qquad ds = \sqrt{1+y'^2}\,dx$$

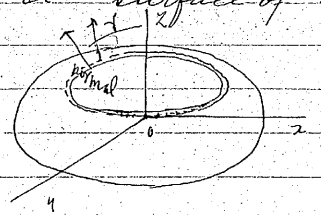
$$= \frac{2\pi}{2}\int_0^x \left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)^2 dx \qquad = \frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)dx = \frac{y}{c}\,dx$$

$$= \frac{2\pi}{c}\int_0^x y^2\,dx$$

$$V = \pi\int_0^x y^2\,dx \qquad V = \frac{c}{2}A.$$

Ex. ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

surface of rev.



$$\cos\gamma = \frac{\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

tangential plane, direct. cos

$$\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}$$

$= \cos\alpha : \cos\beta : \cos\gamma$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

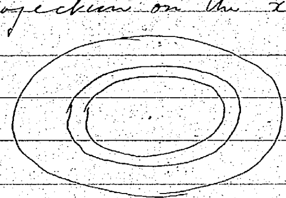
$\cos\gamma$ constant pt. locus

$$\cos\gamma = \frac{\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \lambda$$

$\gamma = 0$, $\cos\gamma = \lambda = 1$ z axis

$\gamma = \frac{1}{2}\pi$, $\cos\gamma = \lambda = 0$ equator on xy plane

projection on the xy plane



area element

$1 : \cos\gamma =$

el. on the surface : el. on the plane

$\cos\gamma$ constant

project

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \frac{\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \lambda$$

eliminate

$$\frac{z^2}{c^4} = \lambda^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)$$

$$\frac{z^2}{c^4}(1-\lambda^2) = \lambda^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

$$\frac{z^2}{c^2} = \frac{c^2\lambda^2}{1-\lambda^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2}\left(1 + \frac{c^2\lambda^2}{a^2(1-\lambda^2)}\right) + \frac{y^2}{b^2}\left(1 + \frac{c^2\lambda^2}{b^2(1-\lambda^2)}\right) = 1$$

projection ellipse

$$a\sqrt{\frac{a^2(1-\lambda^2)}{a^2-(a^2-c^2)\lambda^2}} \qquad b\sqrt{\frac{b^2(1-c^2)}{b^2-(b^2-c^2)\lambda^2}}$$

major axis   minor axis

area.

$$A(\lambda) = \frac{\pi a^2 b^2 (1-\lambda^2)}{\sqrt{(a^2-(a^2-c^2)\lambda^2)(b^2-(b^2-c^2)\lambda^2)}}$$

$$A(\lambda+\Delta\lambda) - A(\lambda) \doteqdot A'(\lambda)\Delta\lambda \text{ or } = A'(\lambda+\theta\Delta\lambda)\Delta\lambda$$

Elementary band on the surf $\dfrac{A'(\lambda+\theta\Delta\lambda)\Delta\lambda}{\lambda}$

$\sum \dfrac{A'(\lambda+\theta\Delta\lambda)\Delta\lambda}{\lambda}$ の 上 の area.

を limit $\quad -\int_0^1 \dfrac{A'(\lambda)}{\lambda}\, d\lambda$

は 計算 … elliptic integral = [illegible]

$$\frac{S}{2} = -\int_0^1 \frac{A'(\lambda)}{\lambda}\, d\lambda$$

$$A(\lambda) = \frac{\pi a^2 b^2 (1-\lambda^2)}{ab\sqrt{\left(1-\frac{a^2-c^2}{a^2}\lambda^2\right)\left(1-\frac{b^2-c^2}{b^2}\lambda^2\right)}}$$

$a^2 > b^2 > c^2$

$$\frac{a^2-c^2}{a^2}\lambda^2 = t^2 = \sin^2\varphi \qquad = \frac{\pi ab\left(1-\frac{a^2}{a^2-c^2}t^2\right)}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$\frac{a^2(b^2-c^2)}{b^2(a^2-c^2)} = k^2 < 1$$

$$\frac{c}{a} = \cos\mu \qquad \sqrt{\frac{a^2-c^2}{a^2}} = \sin\mu$$

$$\frac{c}{b} = \sqrt{1-k^2\sin^2\mu}$$

$$A(\lambda) = \frac{\pi ab(\sin^2\mu - \sin^2\varphi)}{\sin^2\varphi \frac{\cancel{\sin\mu}}{\cos\varphi}\sqrt{1-k^2\sin^2\varphi}} = \omega(\varphi)$$

$$A'(\lambda) = \frac{dA(\lambda)}{d\lambda} \qquad \lambda = \frac{\sin\varphi}{\sin\mu}$$

$$\cancel{A'(\lambda) = \frac{\sin\mu}{\cos\varphi}\frac{d\omega}{d\varphi}}$$

$$\frac{A'(\lambda)}{\lambda}d\lambda = \frac{\sin\mu}{\sin\varphi}\frac{d\omega(\varphi)}{d\varphi}\,d\varphi$$

$$\frac{S}{2} = -\int_0^\mu \frac{\sin\mu}{\sin\varphi}\frac{d\omega}{d\varphi}\,d\varphi = -\sin\mu\int_0^\mu \frac{1}{\sin\varphi}\frac{d\omega}{d\varphi}\,d\varphi$$

$$\frac{1}{\sin\varphi}\frac{d\omega}{d\varphi} = \frac{d}{d\varphi}\left(\frac{\omega}{\sin\varphi}\right) + \frac{\omega\cos\varphi}{\sin^2\varphi}$$

$$= \quad '' \quad + \frac{\pi ab}{\sin^2\mu}\left(\frac{\sin^2\mu}{\sin\varphi\sqrt{1-k^2\sin^2\varphi}} - \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\right)$$

$$\frac{1}{\sin^2\varphi\sqrt{1-k^2\sin^2\varphi}} = \frac{d}{d\varphi}\left(\frac{\cancel{\sqrt{1-k^2\sin^2\varphi}}\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}\sin\varphi}\right) - \frac{k^2\cos^2\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}} \quad = k^2(1-\sin^2\varphi)$$

$$= \quad '' \quad -\frac{1}{\sqrt{1-k^2\sin^2\varphi}} + (1-k^2)\frac{1}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}$$

$$\frac{d}{d\varphi}\left(\frac{\cos\varphi}{\sin\varphi}\sqrt{1-k^2\sin^2\varphi}\right) = -\frac{1}{\sin^2\varphi}\sqrt{1-k^2\sin^2\varphi} + \frac{1}{2}\frac{\cos\varphi\, k^2\sin\varphi\cos\varphi}{\sin\varphi\sqrt{}}$$

$$= -\frac{1-k^2\sin^2\varphi}{\sin^2\varphi\sqrt{}} - \frac{1-k^2\sin^2\varphi-(1-k^2)}{\sqrt{}}$$

$$\frac{1}{\sin^2\varphi\sqrt{1-k^2\sin^2\varphi}} = -\frac{d}{d\varphi}\left(\frac{\cos\varphi}{\sin\varphi}\sqrt{}\right) + \frac{k^2+1-k^2}{\sqrt{}} - \sqrt{1-k^2\sin^2\varphi}$$

$$\frac{S}{2} = -\sin\varphi\int_0^\mu \frac{1}{\sin\varphi}\frac{d\omega}{d\varphi}\,d\varphi = -\sin^2\mu\left\{\left(\frac{\omega}{\sin\varphi}\right)_0^\mu - \frac{\pi ab}{\sin^2\mu}\int_0^\mu\frac{d\varphi}{\sqrt{}}\right.$$

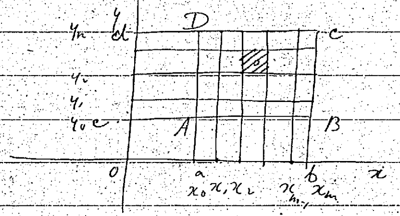
$$\left. + \pi ab\left\{\left(-\frac{\cos\varphi}{\sin\varphi}\sqrt{}\right)_0^\mu + \int_0^\mu\frac{d\varphi}{\sqrt{}} - \int_0^\mu\sqrt{}\,d\varphi\right\}\right\}$$

$\frac{S}{2} = \pi ab \sin^2\mu \cos\mu \sqrt{1-k^2 \sin^2\mu} - \sin^2\mu \left(1 - \frac{\pi ab}{\sin^2\mu}\right) F(\mu, k) + \sin^2\mu\, E(\mu, k)$

[illegible]

Double integral

$f(x, y)$ one valued, continuous in $a \leqq x \leqq b$, $c \leqq y \leqq d$

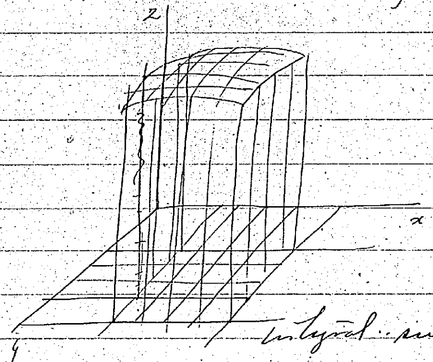


x_i x_{i+1} [illegible] y_k y_{k+1} [illegible] any pt (ξ_{ik}, η_{ik}) [illegible]

$$\sum_{\substack{i=0,m-1\\ k=0,n-1}} f(\xi_{ik}, \eta_{ik}) \underbrace{(x_{i+1} - x_i)(y_{k+1} - y_k)}_{\text{area}}$$

the $\sum$ limit when $m, n \to \infty$, $\Delta x_i, \Delta y_k \to 0$ [illegible] exist [illegible] $\iint_{(R)} f(x, y)\, dx\, dy$ double integral [illegible]

[illegible] geometrical meaning



$z = f(x y)$ continuous surface

$f(\xi, \eta)$ [illegible] ξ, η [illegible] base [illegible] pt [illegible] height [illegible] $f(\xi, \eta)(x_{i+1} - x_i)(y_{k+1} - y_k)$ [illegible] parallel~~p~~iped [illegible] volume [illegible]

integral surface [illegible] vol. [illegible]

$\iint_{(R)} f(x, y)\, dx\, dy$ [illegible] f [illegible] cont [illegible] exist [illegible] definite integral [illegible]

$$\iint f(xy)\, dx\, dy = \int_c^d \left[\int_a^b f(x y)\, dx\right] dy = \int_a^b \left[\int_c^d f(xy)\, dy\right] dx$$

[illegible] double integral [illegible] repeated integral [illegible] repetition [illegible] order [illegible]

$$\sum_{k=0}^{n-1} \sum_{i=0}^{m-1} f(\xi_{ik}, \eta_{ik})(x_{i+1} - x_i)(y_{k+1} - y_k) = \sum_{i=0}^{m-1} (x_{i+1} - x_i)\Big(f(\xi_{i0}, \eta_{i0})(y_1 - y_0)$$

$$+ f(\xi_{i1}, \eta_{i1})(y_2 - y_1) + \cdots + f(\xi_{in-1}, \eta_{in-1})(y_n - y_{n-1})\Big)$$

$\xi_{i0}, \xi_{i1}, \cdots \xi_{in-1} = x_i$ [illegible]

$$= \sum_{i=0}^{m-1} (x_{i+1} - x_i) \int_c^d f(x_i, y)\, dy$$

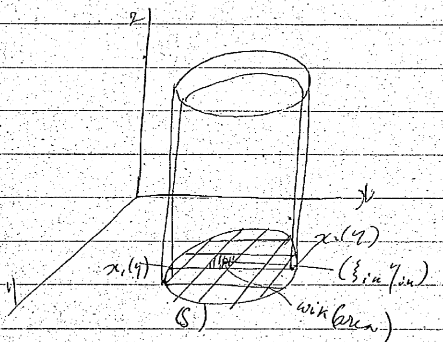
$$\int_c^d f(x_i, y)\, dy = \int_{y_0}^{y_1} + \int_{y_1}^{y_2} + \cdots + \int_{y_{n-1}}^{y_n}$$

$$= f(x_i, \eta_0)(y_1 - y_0) + f(x_i, \eta_1)(y_2 - y_1) + \cdots$$

$$= \int_a^b dx \left(\int_c^d f(xy)\, dy\right)$$

ξ, η [illegible] limit [illegible] ξ, η [illegible]

a, b, c, d [illegible] finite [illegible] repeated integral [illegible] order [illegible] infinite [illegible]



$$\lim \sum f(\xi_{ik}, \eta_{ik})\, w_{ik} = \iint_{(S)} f(x,y)\, dx\, dy$$

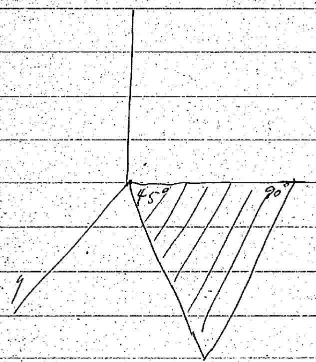
此 limit … (…)

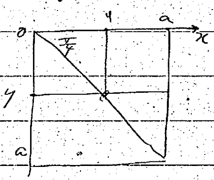
$$= \int_c^d dy \int_{x_1(y)}^{x_2(y)} z\, dx$$

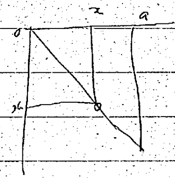
$$= \int_a^b dx \int_{y_1(x)}^{y_2(x)} z\, dy$$

… repeated integral … order … integration … limit …

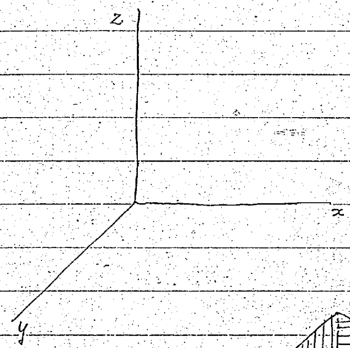
$$\int_0^a dy \int_y^a z\, dx = \int_0^a dx \int_0^x z\, dy$$





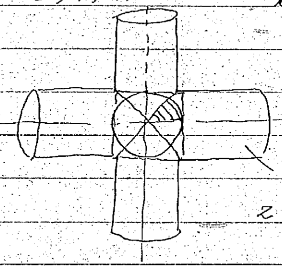


Ex. 1



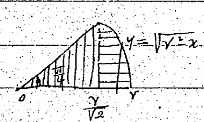
… right circular cylinder … volume …

xy plane … projection …



$y^2 + z^2 = r^2$

$z = \sqrt{r^2 - y^2}$



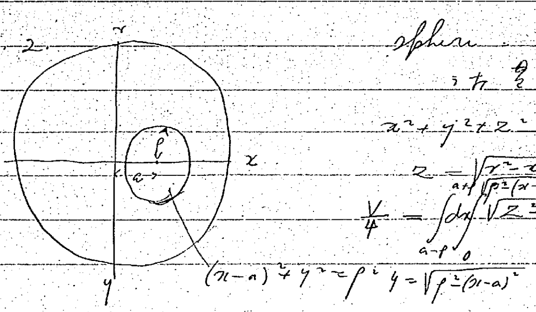
$$\frac{V}{16} = \int_0^{\frac{r}{\sqrt{2}}} dx \int_0^{x} \sqrt{r^2 - y^2}\, dy + \int_{\frac{r}{\sqrt{2}}}^{r} dx \int_0^{\sqrt{r^2 - x^2}} \sqrt{r^2 - y^2}\, dy$$

$$\int \sqrt{r^2 - t^2}\, dt = \frac{1}{2} t\sqrt{r^2 - t^2} + \frac{r^2}{2} \arcsin \frac{t}{r}$$

$$\frac{V}{16} = \int_0^{\frac{r}{\sqrt{2}}} \left\{ \frac{1}{2} x\sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin \frac{x}{r} \right\} dx + \int_{\frac{r}{\sqrt{2}}}^{r} \left\{ \frac{x\sqrt{r^2 - x^2}}{2} + \frac{r^2}{2} \arcsin \frac{\sqrt{r^2 - x^2}}{r} \right\} dx$$

$$= \int_0^r \frac{x\sqrt{r^2 - x^2}}{2}\, dx + \frac{r^2}{2} \int_0^{\frac{r}{\sqrt{2}}} \arcsin \frac{x}{r}\, dx + \frac{r^2}{2} \int_{\frac{r}{\sqrt{2}}}^{r} \arccos \frac{x}{r}\, dx$$

Ex. 2



sphere … right circular cylinder …

$x^2 + y^2 + z^2 = r^2$

$z = \sqrt{r^2 - x^2 - y^2}$

$$\frac{V}{4} = \int_{a-\rho}^{a+\rho} dx \int_0^{\sqrt{\rho^2 - (x-a)^2}} \sqrt{r^2 - x^2 - y^2}\, dy$$

$$\int_0^{\sqrt{\rho^2-(x-a)^2}} \sqrt{r^2-x^2-y^2}\,dy = \left[\frac{y\sqrt{r^2-x^2-y^2}}{2} + \frac{r^2-x^2}{2}\arcsin\frac{y}{\sqrt{r^2-x^2}}\right]_0^{\sqrt{\rho^2-(x-a)^2}}$$

$$= \frac{1}{2}\sqrt{\rho^2-(x-a)^2}\sqrt{r^2-\rho^2+a^2-2ax} + \frac{r^2-x^2}{2}\arcsin\sqrt{\frac{\rho^2-(x-a)^2}{r^2-x^2}}$$

$x-a=t$

$$\int_{a-\rho}^{a+\rho}(\quad)\,dx = \int_{-\rho}^{+\rho}\left\{\frac{1}{2}\sqrt{\rho^2-t^2}\sqrt{r^2-\rho^2-a^2-2at} + \frac{r^2-(t+a)^2}{2}\arcsin\sqrt{\frac{\rho^2-t^2}{r^2-(t+a)^2}}\right\}dt$$

special case [illegible] cylinder ~~z axis~~ radius [illegible] $\rho = \frac{r}{2}$ [illegible]

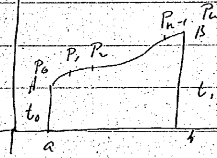
$a = \rho = \frac{r}{2}$

$$\frac{1}{2}\sqrt{2\rho}\int_{-\rho}^{\rho}(\rho-t)\sqrt{\rho+t}\,dt + \int_{-\rho}^{\rho}\frac{4\rho^2-(t+\rho)^2}{2}\arcsin\sqrt{\frac{\rho^2-t^2}{4\rho^2-(t+\rho)^2}}\,dt$$

Curvilinear integrals

$y = f(x)$ one val. & cont. in (a, b)

$$\varphi(x,y) = \varphi(x, f(x)) = \Phi(x)$$



$P_k : (x_k, y_k)$

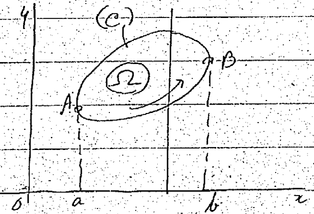
$$\lim \sum_{k=0}^{n-1} \varphi(x_k, y_k)(x_{k+1}-x_k) = \int_{(C)} \varphi(x,y)\,dx$$

$x = x(t)$

$y = y(t)$ [illegible] parameter [illegible]

$$\int_{(C)} \varphi(x,y)\,dx = \int_{t_0}^{t_1} \varphi(x(t), y(t))\,x'(t)\,dt$$

Theorem. [illegible] closed curve [illegible] y axis [illegible] // [illegible] [illegible] curve [illegible] area [illegible] positive direct. [illegible]



$\overset{\frown}{AB}$ $y = y_1$ circle [illegible] $y_1 = -\sqrt{r^2-x^2}$

$\overset{\frown}{AB}$ $y = y_2$ $y_2 = +\sqrt{r^2-x^2}$

$$J = \iint_{(\Omega)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\,dy$$

Ω [illegible] domain

[illegible] $$J = \iint_{(\Omega)}\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)dx\,dy = -\int_{(C)}(P\,dx + Q\,dy)$$

[illegible] <u>Green's theorem</u> [illegible] $P(x,y)$ $Q(x,y)$

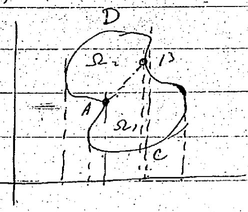
~~Lemma.~~ $$\iint_{(\Omega)}\frac{\partial P}{\partial y}\,dx\,dy = \int_a^b dx\int_{y_1}^{y_2}\frac{\partial P}{\partial y}\,dy = \int_a^b\left(P(x,y_2) - P(x,y_1)\right)dx$$

$$\int_{(C)} P(x,y)\,dx = \int_a^b P(x,y_1)\,dx - \int_a^b P(x,y_2)\,dx$$

[illegible] $$\iint_{(\Omega)}\frac{\partial P}{\partial y}\,dx\,dy = -\int_{(C)} P(x,y)\,dx.$$

hence the theorem.

[illegible] y axis // [illegible]



AB [illegible]

Ω_1 [illegible] Ω_2 [illegible] theorem apply [illegible]

$$\iint_{(\Omega_1)}(\;)\,dx\,dy = -\left\{\int_{ACB} + \int_{BA}\right\}$$

$$\iint_{(\Omega_2)} (\ )\, dx\, dy = -\left\{ \int_{BDA} + \int_{AB} \right\}$$

sum [illegible]

$$\therefore \iint_{(\Omega)} (\ )\, dx\, dy = -\int_{C} (\ )\, d$$

— [illegible] closed curve [illegible] hold [illegible]

[illegible] $P(xy)\, dx + Q(xy)\, dy = d\,F(x, y)$ [illegible]

P, Q [illegible] (~~necess. + suff. condit.~~)

$P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$ [illegible]

[illegible] $P(xy)\, dx + Q\, dy = d\,F(xy)$ [illegible]

$$-\int (P\,dx + Q\,dy) \equiv 0 \qquad F\Big|_{p}^{p}$$

$$\iint_{(\Omega)} \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) dx\, dy \equiv 0$$

[illegible] $\frac{\partial P}{\partial x} \equiv \frac{\partial Q}{\partial y}$ [illegible]

[illegible] pt [illegible] continuous f [illegible]

[illegible] Ω [illegible] domain [illegible]

integral [illegible]

$\therefore \frac{\partial P}{\partial x} \equiv \frac{\partial Q}{\partial y}$ necessary & suff. cond. [illegible]

special case [illegible]

$\frac{\partial P}{\partial y} = 1 \qquad P = y$

$\frac{\partial Q}{\partial x} = -1 \qquad Q = -x \qquad 2\iint_{(\Omega)} dx\, dy = -\int_{C} y\,dx - x\,dy$

$$\iint_{(\Omega)} dx\, dy = \frac{1}{2}\int_{C} (x\,dy - y\,dx)$$

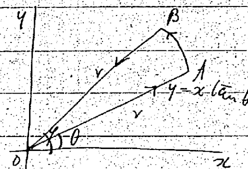
Ω, area [illegible] curvilinear int. [illegible]

Ex. $F(xy) = x^2 y - \frac{y^3}{3}$

$\frac{\partial F}{\partial x} = 2xy = P(xy)$

$\frac{\partial F}{\partial y} = x^2 - y^2 = Q(xy)$

$J_1 = \int_{C} 2xy\, dx$, $\quad J_2 = \int_{C} (x^2 - y^2)\, dy$



C [illegible] OAB [illegible] closed curve

AB [illegible] O [illegible] centre [illegible] circular arc. $OA = OB = r$.

$$\frac{J_1}{2} = \underbrace{\int_0^{r\cos\theta} x \cdot x\tan\theta\, dx}_{\int_{(OA)} xy\,dx} + \underbrace{\int_{r\cos\theta}^{r\cos\varphi} x\cdot\sqrt{r^2 - x^2}\, dx}_{\int_{(AB)} xy\,dx} + \underbrace{\int_{r\cos\varphi}^{0} x\cdot x\tan\varphi\, dx}_{(BO)}$$

$$\frac{J_2}{[illegible]} = \int_0^{r\sin\theta} (y^2\cot^2\theta - y^2)\, dy + \int_{r\sin\theta}^{r\sin\varphi} (r^2 - 2y^2)\, dy + \int_{r\sin\varphi}^{0} (y^2\cot^2\varphi - y^2)\, dy$$

$J_1 + J_2 = 0$ [illegible] verify [illegible]

Transformation of a double integral.

$J = \iint_{(\Omega)} f(x,y)\,dx\,dy$ の variable の transformation により

$x = \varphi(u,v)$ φ, ψ cont. &

$y = \psi(u,v)$ one-val.

とする double integral の 如何に変ずるか

(Ω) は C なる closed curve で 囲まれた domain とする

u, v の 逆に x, y の oneval. & cont. f. として

$u = \Phi(x,y)$ $v = \Psi(x,y)$ Φ, Ψ cont. funct.

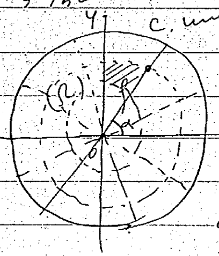
例へば $x = r\cos\theta$, $y = r\sin\theta$ | $r = +\sqrt{x^2+y^2}$, $\theta = \arctan\frac{y}{x}$

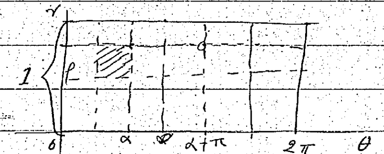
C なる curve は $w(x,y) = 0$ にて表はさる 然らば u, v plane にて如何なる curve となるか

$w(\varphi(u,v), \psi(u,v)) = 0 = W(u,v)$ にて C' curve となり其で囲む doma. (Ω') となる

Ω の中に a pt をとれば 之に対し Ω' 中に之に対する one pt を得ることとなる 逆に Ω' の one pt に対して Ω の one pt を得 pt と pt との one-one corresp. をなす

C. unit circle





concentric circle と // なる straight line と 対応する

一般に Ω の partial doma. に対し Ω' の part. dom. 対応し Jacobian $\frac{\partial(\varphi,\psi)}{\partial(u,v)} \geq 0$ or ≤ 0 in (Ω') same sign をもつとする

Ω の partial dom. $[\omega]_{xy}$ に対する Ω' の $[\omega']_{uv}$

ω と ω' との関係 如何

$A(\omega) = \iint_{(\omega)} dx\,dy$

by Green's theorem $A(\omega) = \int_{(\sigma)} x\,dy$ $\left(\begin{matrix} Q = -x, \\ P = 0 \text{ とし} \end{matrix}\right)$

σ' は σ の posit. sense に対応して $= \pm\int_{(\sigma')} \varphi(u,v)\,d\psi(u,v)$

$= \pm\int_{(\sigma')} \varphi(u,v)\left\{\frac{\partial\psi}{\partial u}du + \frac{\partial\psi}{\partial v}dv\right\}$

此 curvilinear integral を Green's th. により ∬ に直し

$P = \varphi\frac{\partial\psi}{\partial u}$ $Q = \varphi\frac{\partial\psi}{\partial v}$

$= \pm\iint_{(\omega')} \frac{\partial(\varphi,\psi)}{\partial(u,v)}\,du\,dv$

ω' 中にて $\frac{\partial(\varphi,\psi)}{\partial(u,v)}$ の 最大値 M, 最小値 m

$M\iint_{(\omega')} du\,dv \geq \iint_{(\omega')} \frac{\partial(\varphi,\psi)}{\partial(u,v)}\,du\,dv \geq m\iint_{(\omega')} du\,dv$

ω' 中の適当の (ξ, η) に対し $\left[\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right]_{\xi,\eta} A(\omega') = \iint_{(\omega')} \frac{\partial(\varphi,\psi)}{\partial(u,v)}\,du\,dv$

故に $A(\omega) = \left|\left[\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right]_{\xi,\eta}\right| A(\omega')$

$$\iint_{(\Omega)} f(x,y)\,dx\,dy = \lim \sum f(\xi_i,\eta_i)\,A(\omega_i)$$

$$= \lim \sum f(\varphi(\xi_i',\eta_i'),\psi(\xi_i',\eta_i'))\,A(\omega_i)$$

$x = \xi_i \quad u = \xi_i'$

$y = \eta_i \quad v = \eta_i'$

$$= \lim \sum f(\qquad)\left|\left(\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right)_{u=\xi_i',\,v=\eta_i'}\right| A(\omega_i')$$

$$= \iint_{(\Omega')} f(\varphi,\psi)\left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right| du\,dv$$

[illegible] Goursat [illegible] elegant [illegible]

[illegible] Green's theorem [illegible]

$x = r\cos\theta \qquad \dfrac{\partial(\varphi,\psi)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$

$y = r\sin\theta$

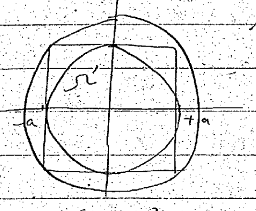
$$\iint_{(\Omega)} f(x,y)\,dx\,dy = \iint_{(\Omega')} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

specially $f = 1$

$$\iint_{(\Omega)} dx\,dy = \iint_{(\Omega')} r\,dr\,d\theta \qquad \text{area formula}$$

2 $z = f(x,y)$ [illegible] xy plain [illegible] project [illegible] Ω [illegible] surface [illegible] volume [illegible] x, y [illegible] polar coordinate [illegible] z [illegible] cylindrical coordinates [illegible] vol. [illegible] $\iint_{(\Omega')} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta$ [illegible]

Ex. $\iint_{(\Omega)} e^{-x^2-y^2}\,dx\,dy = \iint_{(\Omega')} e^{-r^2} r\,dr\,d\theta$



$$左辺 = \int_{-a}^{+a} dx \int_{-a}^{+a} e^{-x^2-y^2}\,dy$$

$$= \int_{-a}^{+a} e^{-x^2}dx \int_{-a}^{+a} e^{-y^2}dy$$

$$= \left\{\int_{-a}^{+a} e^{-x^2}dx\right\}^2$$

$$\iint e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R \frac{e^{-r^2}}{2}\,d(r^2) = 2\pi\left(-\frac{e^{-r^2}}{2}\right)_0^R$$

$$= \pi(1 - e^{-R^2})$$

Ω' [illegible] a [illegible] rad. [illegible] circle [illegible]

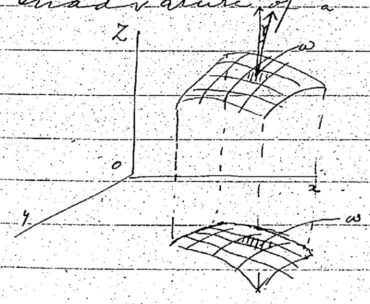
" $\sqrt{2}\,a$ [illegible]

$R \to +\infty$ [illegible] $e^{-R^2} = 0$ $R \to \infty$

$$故に \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \pi \quad = 2\int_0^\infty e^{-x^2}dx$$

Gauss's error function

Quadrature of a curve



ω [illegible] plane [illegible] plane [illegible] project

[illegible] area [illegible] ratio [illegible]

$\omega\cos\gamma = \omega'$ direction cosine

$\dfrac{\partial z}{\partial x} : \dfrac{\partial z}{\partial y} : -1 = \cos\alpha : \cos\beta : \cos\gamma = p : q : -1$

$$\cos\gamma = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} = \frac{1}{\sqrt{1+p^2+q^2}}$$

$$\omega = \frac{\omega'}{\cos\gamma} = \omega'\sqrt{1+p^2+q^2}$$

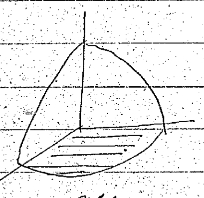
$$A = \sum \omega = \sum \omega' \sqrt{1+p^2+q^2} = \iint_{(D)} \sqrt{1+p^2+q^2}\, dx\, dy$$

Ex. $x^2+y^2+z^2=r^2$

$$p = \frac{\partial z}{\partial x} = -\frac{x}{z} \qquad q = \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2} = \frac{\sqrt{x^2+y^2+z^2}}{z} = \frac{r}{z}$$

$$\frac{A}{8} = \iint \frac{r}{\sqrt{r^2-x^2-y^2}}\, dx\, dy = r\int_0^r dx \int_0^{\sqrt{r^2-x^2}} \frac{dy}{\sqrt{r^2-x^2-y^2}}$$



Ellipsoid [illegible]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad p = -\frac{c^2x}{a^2z}, \quad q = -\frac{c^2y}{b^2z}$$

$$\sqrt{1+p^2+q^2} = \frac{c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}{z} = \frac{c\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{1}{c^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

[illegible] surface of revolution [illegible] treat [illegible]

$$A = \iint \sqrt{1+p^2+q^2}\, dx\, dy$$ cylindrical coord. [illegible]

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$A = \iint \sqrt{1+p^2+q^2}\, r\, d\theta\, dr$$

$$z = f(x,y) = F(r,\theta)$$

$\cos\theta$ $\qquad \frac{\partial z}{\partial r} = p\cos\theta + q\sin\theta$

$-\sin\theta$ $\qquad \frac{1}{r}\frac{\partial z}{\partial \theta} = -p\sin\theta + q\cos\theta$

$$\frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta = p$$

$\sin\theta$ $\quad$ $+\cos\theta$

$$\frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta = q$$

$$p^2+q^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$A = \iint \sqrt{1+\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\, r\, dr\, d\theta$$

$$= \iint \sqrt{r^2 + r^2\left(\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2}\, dr\, d\theta$$

Ex. $2z = \cancel{ax^2 - by^2}\ \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Hyperbolic paraboloid

$$p = \frac{x}{a^2}, \quad q = -\frac{y}{b^2}, \quad \sqrt{1+p^2+q^2} = \sqrt{1+\frac{x^2}{a^4}+\frac{y^2}{b^4}}$$

[illegible] $x^2+y^2=r^2$ [illegible] intersect [illegible] area

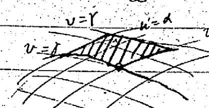
$$\int_0^r dx \int_0^{\sqrt{r^2-x^2}} \sqrt{1+\frac{x^2}{a^4}+\frac{y^2}{b^4}}\, dx\, dy$$

Hyperbolic Paraboloid [illegible] two systems of generating lines [illegible] any two generating lines [illegible] space Vareek quadrat [illegible] area [illegible]

$$\begin{cases} \frac{x}{a}+\frac{y}{b} = u \\ 2z = u\left(\frac{x}{a}-\frac{y}{b}\right) \end{cases} \qquad \begin{cases} \frac{x}{a}-\frac{y}{b} = v \\ 2z = v\left(\frac{x}{a}+\frac{y}{b}\right) \end{cases}$$

two systems of generating lines.

$$\frac{x}{a} = \frac{u+v}{2} \qquad \frac{y}{b} = \frac{u-v}{2} \qquad z = \frac{uv}{2}$$



$u=\alpha,\ u=\beta$

$v=\gamma,\ v=\delta$

$$\sqrt{1+p^2+q^2} = \sqrt{1+\frac{x^2}{a^4}+\frac{y^2}{b^4}} = \sqrt{1+\frac{1}{a^2}\left(\frac{u+v}{2}\right)^2+\frac{1}{b^2}\left(\frac{u-v}{2}\right)^2} = \cancel{\sqrt{1+\frac{u^2+v^2}{2}}}$$

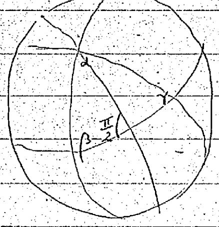
$$\frac{\partial(\varphi,\psi)}{\partial(u,v)} = \begin{vmatrix} \frac{a}{2} & \frac{a}{2} \\ \frac{b}{2} & -\frac{b}{2} \end{vmatrix} = \frac{ab}{4}(-2) = -\frac{ab}{2}$$

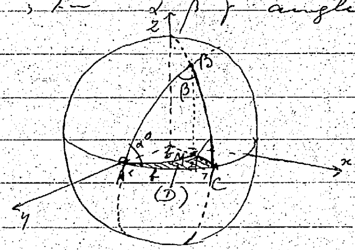
$$\therefore\ x = \frac{a}{2}(u+v) \qquad y = \frac{b}{2}(u-v)$$

$$\iint \sqrt{1+p^2+q^2}\,dx\,dy = \frac{ab}{2}\int_\alpha^\beta du\int_\gamma^\delta \sqrt{1+\cancel{\frac{u^2+v^2}{2}}\,\frac{1}{a^2}\left(\frac{u+v}{2}\right)^2+\frac{1}{b^2}\left(\frac{u-v}{2}\right)^2}\,dv$$

Ex. area of a spherical triangle

unit circle, r = ... $\alpha\ \beta\ \gamma$ angle $\quad A = \alpha+\beta+\gamma-\pi$

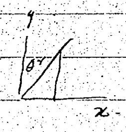




$$x^2+y^2+z^2=1$$

$$p = -\frac{x}{z},\ q = \frac{y}{z}$$

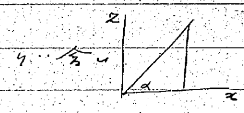
$$\sqrt{1+p^2+q^2} = \frac{\sqrt{x^2+y^2+z^2}}{z} = \frac{1}{z}$$



$x = r\sin\theta$ ... cylindrical coord. ...

$y = r\cos\theta$

area ABC $= \iint_{(D)} \frac{r\,dr\,d\theta}{\sqrt{1-r^2}}$

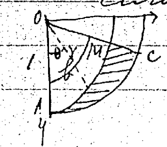


plane AOB $\quad z = \tan\alpha \cdot x$

curve AM $\quad \left\{ x^2+y^2+z^2=1 \right.$

$\qquad x^2+y^2+\tan^2\alpha\, x^2 = 1$

$\qquad y^2+\sec^2\alpha\, x^2 = 1$ ellipse



polar coord. ... $r^2\sec^2\alpha\sin^2\theta + r^2\cos^2\theta = 1$

$$r^2 = \frac{1}{\cos^2\theta+\sin^2\theta\sec^2\alpha}$$

$$\rho = \frac{\cos\alpha}{\sqrt{\sin^2\theta+\cos^2\alpha\cos^2\theta}} \qquad \text{ellipse AM}$$

$$\therefore\ \text{area ABC} = \iint_D \frac{r\,dr\,d\theta}{\sqrt{1-r^2}} = \int_0^b d\theta\int_\rho^1 \frac{r\,dr}{\sqrt{1-r^2}}$$

$$= \int_0^b \left(-\sqrt{1-r^2}\right)_\rho^1 d\theta = \int_0^b \sqrt{1-\rho^2}\,d\theta$$

$$= \int_0^b \frac{\sin\alpha\sin\theta\,d\theta}{\sqrt{\sin^2\theta+\cos^2\alpha\cos^2\theta}} = \int_0^b \frac{\sin\alpha\sin\theta\,d\theta}{\sqrt{1-\sin^2\alpha\cos^2\theta}}$$

$$= -\left[\arcsin(\sin\alpha\cos\theta)\right]_0^b$$

$$= \alpha - \arcsin(\sin\alpha\cos b)$$

Spherical trig. により $\sin\alpha\cos b = \cos\beta = \sin\left(\frac{\pi}{2}-\beta\right)$

$$= \alpha - \left(\frac{\pi}{2}-\beta\right) = \alpha+\beta-\frac{\pi}{2}$$

他の二つの area ...

$A+B+C$ ... area

$= \alpha+\beta+\gamma-\pi$ は spherical excess

... surface の parameter による表示 ...

Transformation

$$x = f(u,v) \quad y = \varphi(u,v) \quad z = \psi(u,v)$$

$$\frac{\partial z}{\partial u} = p\frac{\partial x}{\partial u}+q\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = p\frac{\partial x}{\partial v}+q\frac{\partial y}{\partial v}.$$

$$\therefore\ p = \frac{-\dfrac{\partial(yz)}{\partial(uv)}}{\dfrac{\partial(xy)}{\partial(uv)}} \qquad q = \frac{-\dfrac{\partial(zx)}{\partial(uv)}}{\dfrac{\partial(xy)}{\partial(uv)}}$$

$$1+p^2+q^2 = \frac{1}{\left[\dfrac{\partial(xy)}{\partial(uv)}\right]^2}\left\{\left[\frac{\partial(yz)}{\partial(uv)}\right]^2+\left[\frac{\partial(zx)}{\partial(uv)}\right]^2+\left[\frac{\partial(xy)}{\partial(uv)}\right]^2\right\}$$

$$\sqrt{1+p^2+q^2} = \frac{1}{\left|\dfrac{\partial(xy)}{\partial(uv)}\right|}\left\{\qquad\right\}^{\frac{1}{2}}$$

$$dx\,dy = \left|\frac{\partial(xy)}{\partial(uv)}\right| du\,dv$$

$$\iint\sqrt{1+p^2+q^2}\,dx\,dy = \iint\left\{\left[\frac{\partial(yz)}{\partial(uv)}\right]^2+\left[\frac{\partial(zx)}{\partial(uv)}\right]^2+\left[\frac{\partial(xy)}{\partial(uv)}\right]^2\right\}^{\frac{1}{2}} du\,dv$$

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2 = (bc'-b'c)^2 + (ca'-ac')^2+(ab'-a'b)^2$$

$$(\underbrace{x_u^2+y_u^2+z_u^2}_{E})(\underbrace{x_v^2+y_v^2+z_v^2}_{G})-(\underbrace{x_ux_v+y_uy_v+z_uz_v}_{F})^2$$

differential form [illegible]

$$\therefore \iint\sqrt{EG-F^2}\,du\,dv$$ most general formula for transformation

Especially $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi \qquad r = r(\theta\varphi)$

$z = r\cos\theta$

$$E = r^2+\left(\frac{\partial r}{\partial\theta}\right)^2, \quad F = \frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\varphi}, \quad G = r^2\sin^2\theta+\left(\frac{\partial r}{\partial\varphi}\right)^2$$

$$\therefore\ EF-G^2 = r^2\left(\left(r^2+\left(\frac{\partial r}{\partial\theta}\right)^2\right)\sin^2\theta+\left(\frac{\partial r}{\partial\varphi}\right)^2\right)$$

sphere [illegible] $r = $ Const. ~~Elliptic coordinate~~

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ [illegible] Ellipsoid [illegible] $x = a\sin\theta\cos\varphi$

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \quad (a^2>b^2>c^2) \qquad y = b\sin\theta\sin\varphi$$

$z = c\cos\theta$

λ [illegible] single quadric [illegible] (confocal quadric)

[illegible] $x\ y\ z$ [illegible] pass [illegible] quadric [illegible] λ [illegible]

[illegible] λ [illegible] 3rd degree [illegible] roots [illegible]

$\lambda_1\ \lambda_2\ \lambda_3$ [illegible] roots [illegible]

$a^2>\lambda_1>b^2>\lambda_2>c^2>\lambda_3$

λ_1 Hyperboloid of two sheets

λ_2 〃 one sheet

λ_3 Ellipsoid

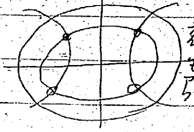
[illegible] $\lambda_1\ \lambda_2\ \lambda_3$ [illegible] point eq. [illegible]

$$x^2 = \frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2-\lambda_1)(b^2-\lambda_2)(b^2-\lambda_3)}{(b^2-c^2)(b^2-a^2)}$$

$$z^2 = \frac{(c^2-\lambda_1)(c^2-\lambda_2)(c^2-\lambda_3)}{(c^2-a^2)(c^2-b^2)}$$

[illegible] Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ [illegible] $\lambda_3=0$ [illegible] equation



λ_3 [illegible] 0 [illegible] $\lambda_1\ \lambda_2$ [illegible] ellipsoid

[illegible] pts [illegible]

$x^2\ y^2\ z^2$ [illegible]

其ノ十八ニ在ルニ $\frac{1}{8}$, Ellipsoid ノ上ノ pt ヲ表ハス

$$x = \frac{a\sqrt{(a^2-\lambda_1)(a^2-\lambda_2)}}{\sqrt{(a^2-b^2)(a^2-c^2)}}$$

Elliptic coordinates

$$y = \frac{b\sqrt{-(b^2-\lambda_1)(b^2-\lambda_2)}}{\sqrt{(a^2-b^2)(b^2-c^2)}} \qquad \left(\begin{array}{l} a^2 \geqq \lambda_1 \geqq b^2 \\ b^2 \geqq \lambda_2 \geqq c^2 \end{array}\right)$$

$$z = \frac{c\sqrt{(c^2-\lambda_1)(c^2-\lambda_2)}}{\sqrt{(c^2-a^2)(c^2-b^2)}}$$

$\lambda_1 = u$, $\lambda_2 = v$ トスル

$$F(\lambda) \equiv (a^2-\lambda)(b^2-\lambda)(c^2-\lambda)$$

$$A \equiv (b^2-c^2)(c^2-a^2)(a^2-b^2) \text{ トスル}$$

$$E = -\frac{1}{4AF(\lambda_1)}\left\{\sum a^4(b^2-c^2)\lambda_1^2 + \sum a^2(b^4-c^4)\lambda_1\lambda_2\right\}$$

$$= \frac{\lambda_1(\lambda_1-\lambda_2)}{4F(\lambda_1)}$$

$$G = \frac{\lambda_2(\lambda_2-\lambda_1)}{4F(\lambda_2)}$$

$$F = 0$$

$$EG - F^2 = \frac{-\lambda_1\lambda_2(\lambda_1-\lambda_2)^2}{16F(\lambda_1)F(\lambda_2)}$$

$$\iint\sqrt{EG-F^2}\,d\lambda_1 d\lambda_2 = \iint \frac{\lambda_1-\lambda_2}{4}\,\frac{\sqrt{\lambda_1\lambda_2}\,d\lambda_1 d\lambda_2}{\sqrt{-F(\lambda_1)F(\lambda_2)}}$$

$$\sqrt{F(\lambda_1)}\sqrt{-F(\lambda_2)}$$

Elliptic integral

故ニ area ヲ計算スル… 之ハ surface element ヲ 寄セ…用フ $d\sigma = \sqrt{EG-F^2}\,d\lambda_1 d\lambda_2$

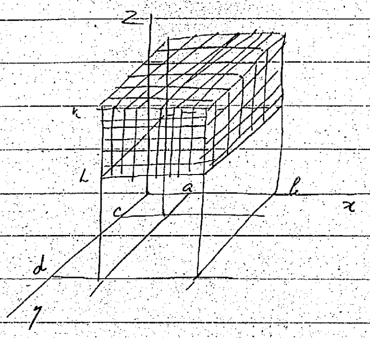
Multiple integrals.

同様ニ Triple integral ヲ定義

(x,y,z) $\quad a \geqq x \geqq b$

parallelepiped ヲ考ヘ $\quad c \geqq y \geqq d$

$h \geqq z \geqq k$



$x_i \quad i = 1, 2, \ldots, l-1$

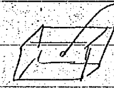
$y_j \quad j = 1, 2, \ldots, m-1$

$z_k \quad k = 1, 2, \ldots, n-1$

$x_i \geqq x \geqq x_{i-1}$

$y_j \geqq y \geqq y_{j-1}$

$z_k \geqq z \geqq z_{k-1}$



$\xi_{ijk}\ \eta_{ijk}\ \zeta_{ijk}$

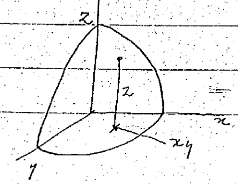
$$\Delta v_{ijk} = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

$$\lim \sum_{ijk} F(\xi_{ijk}\ \eta_{ijk}\ \zeta_{ijk})\,\Delta v_{ijk} = \iiint F(x y z)\,dx\,dy\,dz.$$

一般ニ three dimensional domain D ニテ $f(x,y,z)$ ノ

$$\lim \sum f(\xi_{ijk}\ \eta_{ijk}\ \zeta_{ijk})\,\Delta x \Delta y \Delta z = \iiint_{(D)} f(x y z)\,dx\,dy\,dz \text{ トスル}$$

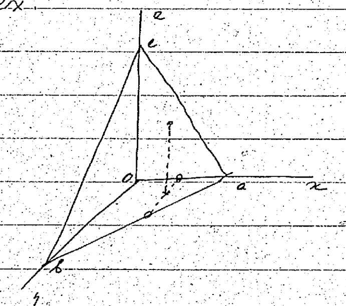
之ハ repeated integral ニ reduce サレル



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

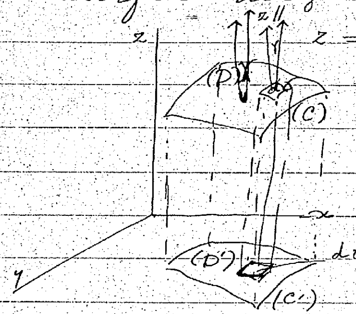
$$\iiint_{(D)} f(x y z)\,dx\,dy\,dz = \int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy \left[\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x y z)\,dz\right]$$

Ex. $x=0,\ y=0,\ z=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ plane ... Tetrahedron



$$\iiint f\,dx\,dy\,dz = \int_0^a dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} f\,dz$$

Surface integrals & Green's theorem.



$z=\varphi(x,y)$ surface

(C) contour, curve

z axis ... one pt ...

... surface ...

normal ... direct.

$C(x,y,z)$ given funct.

$z=\varphi(x,y)$

$$\iint_{(D')} C(xyz)\,dx\,dy \qquad z=\varphi(xy)$$

$$\cos\gamma\, d\sigma = d\sigma' = dx\,dy$$

$$= \iint_{(D)} \underbrace{C(xyz)\cos\gamma\, d\sigma}_{\text{surface pt funct}} \qquad \text{Surface integral}$$

$A(x,y,z)$ ... yz plane project

$$\iint_{(D'')} A(xyz)\,dy\,dz = \iint_{(D)} A(xyz)\cos\alpha\, d\sigma$$

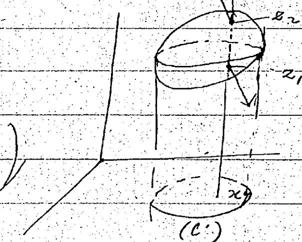
$$\iint_{(D''')} B(xyz)\,dz\,dx = \iint_{(D)} B(xyz)\cos\beta\, d\sigma$$

$\cos\alpha,\ \cos\beta,\ \cos\gamma$ ... surface normal direction cosine

$$\iint_{(D)} (A\cos\alpha + B\cos\beta + C\cos\gamma)\,d\sigma \quad \text{... surface integral}$$

$$J = \iiint_{(\Omega)} \left(\frac{\partial A(xyz)}{\partial x} + \frac{\partial B(xyz)}{\partial y} + \frac{\partial C(xyz)}{\partial z}\right) dx\,dy\,dz$$

Ω solid ... x, y, z axis // ... only two pts



$$\iiint_{(\Omega)} \frac{\partial C}{\partial z}\,dx\,dy\,dz$$

$$= \iint_{(C')} dx\,dy \left(C(xyz_2) - C(xy,z_1)\right)$$

$$\iint C(xyz)\,dx\,dy = \iint C(xyz_2)\,dxdy - \iint C(xyz_1)\,dx\,dy$$

$$= \iint_{(S)} C(xyz)\,dx\,dy$$

$$J = \iint_{(S)} (A\,dy\,dz + B\,dz\,dx + C\,dx\,dy)$$

Green's theorem

一般ニ axis // straight line ト二ヶ以上ニ交ハ surface ニ拡張セラル

normal ヲ引ク [illegible] surface モアル (Möbius Blatt)

今 $A = x,\ B = y,\ C = z$ ト置ケバ

$$3\iiint_{(\Omega)} dx\,dy\,dz = \iint_{(S)} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

又 $$J = \iiint_{(\Omega)} \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right) dx\,dy\,dz$$

$U(x, y, z)\quad V(x, y, z)$

$$\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial x}\right) - U\frac{\partial^2 V}{\partial x^2}$$

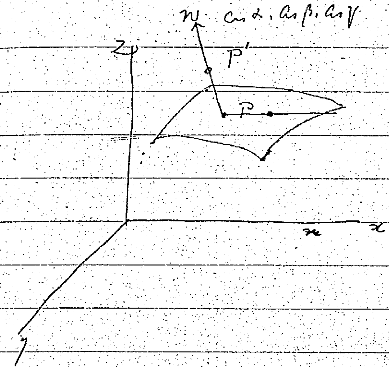
$$J = \iiint_{(\Omega)} \left(\frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(U\frac{\partial V}{\partial z}\right) \right) dx\,dy\,dz$$

$$- \iiint_{(\Omega)} U\Delta(V)\,dx\,dy\,dz \qquad \Delta(V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \iint_{(S)} U\left(\frac{\partial V}{\partial x}\underbrace{dy\,dz}_{\cos\alpha\, d\sigma} + \frac{\partial V}{\partial y}dz\,dx + \frac{\partial V}{\partial z}dx\,dy\right) - \iiint_{(\Omega)} U\Delta(V)\,dx\,dy\,dz$$

$$= \iint_{(S)} U\left(\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\cos\beta + \frac{\partial V}{\partial z}\cos\gamma\right) d\sigma - \iiint_{(\Omega)} U\Delta(V)\,dx\,dy\,dz$$

$$= \iint_{(S)} U\frac{\partial V}{\partial n}\,d\sigma - \iiint_{(\Omega)} U\Delta(V)\,dx\,dy\,dz$$



$P : x, y, z$

$P' : x', y', z'$

$$\frac{V_{P'} - V_P}{P'P} = \frac{V(x'y'z') - V(xyz)}{P'P}$$

$P'P = \Delta n$

$x' = x + \Delta n \cos\alpha$

$y' = y + \Delta n \cos\beta$

$z' = z + \Delta n \cos\gamma$

$$\frac{V_{P'} - V_P}{P'P} = \frac{V(x + \Delta n\cos\alpha,\ y + \Delta n\cos\beta,\ z + \Delta n\cos\gamma) - V(x, y, z)}{\Delta n}$$

$$= \frac{\cos\alpha\,\Delta n\,\frac{\partial V}{\partial x} + \cos\beta\,\Delta n\,\frac{\partial V}{\partial y} + \cos\gamma\,\Delta n\,\frac{\partial V}{\partial z} + (\Delta n)_2}{\Delta n}$$

$$= \cos\alpha\frac{\partial V}{\partial x} + \cos\beta\frac{\partial V}{\partial y} + \cos\gamma\frac{\partial V}{\partial z} + (\Delta n)$$

此 $$\lim_{\Delta n \to 0} \frac{V_{P'} - V_P}{P'P} = \frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\cos\beta + \frac{\partial V}{\partial z}\cos\gamma = \frac{\partial V}{\partial n}$$

n ハ normal direct.

differentiate [illegible]

sphere ナレバ n ハ radius vector ニシテ $\frac{\partial V}{\partial r}$

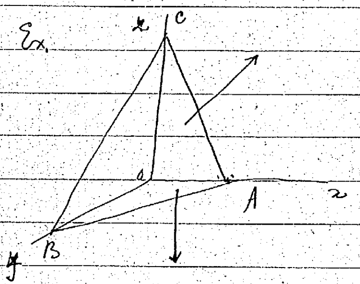
J ハ U, V ニ対シ symmetrical ナル

$$J = \iint_{(S)} V\left(\frac{\partial U}{\partial n}\right) d\sigma - \iiint_{(\Omega)} V\Delta U\,dx\,dy\,dz \quad [illegible]$$

故ニ $$\iiint_{(\Omega)} (U\Delta V - V\Delta U)\,dx\,dy\,dz = \iint_{(S)} \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) d\sigma$$

是ヲ Green's theorem ト云フ [illegible]

Ex.



$x=0,\ y=0,\ z=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$J=\iint_{(S)}(x^2+y^2+z^2)\,dx\,dy$$

$$=\iint_{(AOB)}+\cancel{\iint_{(BOC)}}+\cancel{\iint_{(COA)}}+\iint_{(ABC)}$$

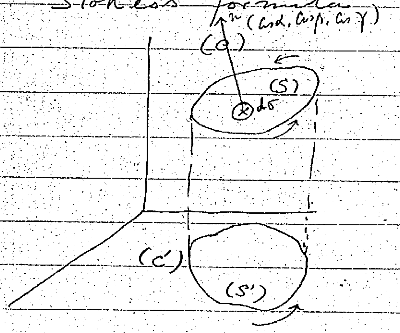
$$=-\iint_{(AOB)}(x^2+y^2)\,dx\,dy+\iint_{(OAB)}\left(x^2+y^2+c^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2\right)dx\,dy$$

$$=c^2\iint_{(AOB)}\left(1-\frac{x}{a}-\frac{y}{b}\right)^2dx\,dy$$

$$=\frac{abc^2}{12}$$

[illegible] $\iint_{(S)}\sum x^2\,dx\,dy+\iint_{(S)}\sum x^2\,dy\,dz+\iint_{(S)}\sum x^2\,dz\,dx=\frac{abc(a+b+c)}{12}$

Stokes's formula



(C) closed curve in space [illegible]
(S) [illegible] surface $z=\varphi(x,y)$ [illegible]
[illegible] z axis [illegible]
[illegible]
C' positive direction
[illegible] C [illegible]
surface normal [illegible]
positive direction [illegible]

[illegible] positive normal direction [illegible]
$A(x,y,z)$ continuous function [illegible] x, y, z [illegible]
(C) curve [illegible] point [illegible] curve [illegible] integral

$$\int_{(C)}A(x,y,z)\,dx=\int A(x,y(x),z(x))\,dx\qquad\begin{cases}z=z(x)\\y=y(x)\end{cases}$$

$$=\int_{(C')}A(x,y,\varphi(x,y))\,dx$$

[illegible] $\iint_{(S')}\frac{\partial P}{\partial y}\,dx\,dy=-\int_{(C')}P\,dx$

$$P(x,y)\equiv A(x,y,\varphi(x,y))$$

$$\frac{\partial P}{\partial y}=\frac{\partial A}{\partial y}+\frac{\partial A}{\partial z}\frac{\partial z}{\partial y}$$

$$\cos\alpha:\cos\beta:\cos\gamma=\frac{\partial z}{\partial x}:\frac{\partial z}{\partial y}:-1$$

$$\frac{\partial P}{\partial y}=\frac{\partial A}{\partial y}-\frac{\cos\beta}{\cos\gamma}\frac{\partial A}{\partial z}=\frac{1}{\cos\gamma}\left(\cos\gamma\frac{\partial A}{\partial y}-\cos\beta\frac{\partial A}{\partial z}\right)$$

$$\int_{(C)}A(x,y,z)\,dx=-\iint_{(S')}\left(\cos\gamma\frac{\partial A}{\partial y}-\cos\beta\frac{\partial A}{\partial z}\right)\frac{dx\,dy}{\cos\gamma}$$

$$\cos\gamma\,d\sigma=dx\,dy$$

$$\int_{(C)}A(x,y,z)\,dx=\iint_{(S)}\left(\cos\beta\frac{\partial A}{\partial z}-\cos\gamma\frac{\partial A}{\partial y}\right)d\sigma$$

[illegible] space [illegible] curvilinear integral [illegible] surface integral [illegible]

$$\int_{(C)}B(x,y,z)\,dy=\iint_{(S)}\left(\cos\gamma\frac{\partial B}{\partial x}-\cos\alpha\frac{\partial B}{\partial z}\right)d\sigma$$

$$\int_{(C)} C(x,y,z)\,dz = \iint_{(S)} \left(\cos\alpha \frac{\partial C}{\partial y} - \cos\beta \frac{\partial C}{\partial x}\right) d\sigma$$

— [illegible]

$$\int_{(C)} (A\,dx + B\,dy + C\,dz) = \iint_{(S)} \left\{ \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)\cos\alpha + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)\cos\beta + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\cos\gamma \right\} d\sigma$$

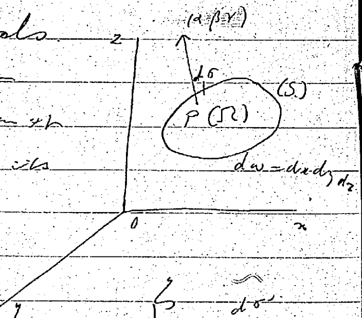
or
$$\int_{(C)} A\,dx + B\,dy + C\,dz = \iint_{(S)} \left\{ \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy\,dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz\,dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy \right\}$$

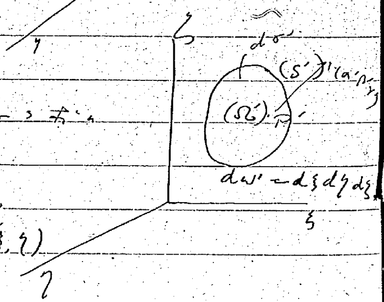
is Stokes' formula [illegible]

Transformation of triple integrals

$x = f(\xi\,\eta\,\zeta)$ one valued +
$y = \varphi(\xi\,\eta\,\zeta)$ continuous
$z = \psi(\xi\,\eta\,\zeta)$ $\xi\,\eta\,\zeta$ solve

solid Ω point, Ω' point
one to one correspondence
(S), (S') correspond
surface element $d\sigma$ $d\sigma'$
volume element $d\omega$ $d\omega'$ [illegible]
$\cos\gamma\, d\sigma = dx\,dy$
$\cos\gamma'\, d\sigma' = d\xi\,d\eta$
$x = f(\xi\,\eta\,\zeta)$ $y = \varphi(\xi\,\eta\,\zeta)$ $\zeta = \zeta(\xi,\eta)$





$\therefore\ x = F(\xi\,\eta)$
$\quad y = \Phi(\xi\,\eta)$

$$dx\,dy = \frac{\partial(F,\Phi)}{\partial(\xi\,\eta)}\,d\xi\,d\eta$$ plane transformation

$$\frac{\partial(F,\Phi)}{\partial(\xi\,\eta)} = \frac{\partial F}{\partial \xi}\frac{\partial \Phi}{\partial \eta} - \frac{\partial F}{\partial \eta}\frac{\partial \Phi}{\partial \xi}$$

$$= \left(\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial \xi}\right)\left(\frac{\partial \varphi}{\partial \eta} + \frac{\partial \varphi}{\partial \zeta}\frac{\partial \zeta}{\partial \eta}\right) - \left(\frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial \eta}\right)\left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \zeta}\frac{\partial \zeta}{\partial \xi}\right)$$

$$= \frac{\partial(f,\varphi)}{\partial(\xi\,\eta)} + \frac{\partial \zeta}{\partial \eta}\frac{\partial(f,\varphi)}{\partial(\xi\,\zeta)} + \frac{\partial \zeta}{\partial \xi}\frac{\partial(f,\varphi)}{\partial(\zeta\,\eta)}$$

$$= \frac{\partial(f,\varphi)}{\partial(\xi\,\eta)} - \frac{\partial \zeta}{\partial \eta}\frac{\partial(f,\varphi)}{\partial(\zeta\,\xi)} - \frac{\partial \zeta}{\partial \xi}\frac{\partial(f,\varphi)}{\partial(\eta\,\zeta)}$$

$$\frac{\partial \zeta}{\partial \xi} : \frac{\partial \zeta}{\partial \eta} : -1 = \cos\alpha' : \cos\beta' : \cos\gamma'$$

$$\cos\gamma\, d\sigma = \left\{ \frac{\partial(f\,\varphi)}{\partial(\eta\,\zeta)}\cos\alpha' + \frac{\partial(f\,\varphi)}{\partial(\zeta\,\xi)}\cos\beta' + \frac{\partial(f\,\varphi)}{\partial(\xi\,\eta)}\cos\gamma' \right\} d\sigma'$$

$$V = \iint_{(S)} z\,dx\,dy = \iint_{(S)} z\cos\gamma\, d\sigma = \iint \psi(\xi\,\eta\,\zeta)\cos\gamma\, d\sigma$$

$$= \iint_{(S')} \psi(\xi\,\eta\,\zeta)\left\{ \frac{\partial(f\,\varphi)}{\partial(\eta\,\zeta)}\cos\alpha' + \frac{\partial(f\,\varphi)}{\partial(\zeta\,\xi)}\cos\beta' + \frac{\partial(f\,\varphi)}{\partial(\xi\,\eta)}\cos\gamma' \right\} d\sigma'$$

$$= \iint_{(S')} \left\{ \psi \frac{\partial(f\,\varphi)}{\partial(\eta\,\zeta)}\,d\eta\,d\zeta + \psi \frac{\partial(f\,\varphi)}{\partial(\zeta\,\xi)}\,d\zeta\,d\xi + \psi \frac{\partial(f\,\varphi)}{\partial(\xi\,\eta)}\,d\xi\,d\eta \right\}$$

by Green's theorem

$$= \iiint_{(\Omega')} \left\{ \frac{\partial}{\partial\xi}\left(\psi \frac{\partial(f\varphi)}{\partial(\eta\zeta)}\right) + \frac{\partial}{\partial\eta}\left(\psi\frac{\partial(f\varphi)}{\partial(\zeta\xi)}\right) + \frac{\partial}{\partial\zeta}\left(\psi\frac{\partial(f\varphi)}{\partial(\xi\eta)}\right)\right\} d\xi\, d\eta\, d\zeta$$

$$= \iiint_{(\Omega')} \frac{\partial(f\varphi\psi)}{\partial(\xi\eta\zeta)}\, d\xi\, d\eta\, d\zeta$$

$$= \iiint_{(\Omega)} dx\, dy\, dz$$

$$J = \iiint_{(\Omega)} F(x, y, z)\, dx\, dy\, dz$$

element ニツイテ.. $\iiint_{(\omega')} \frac{\partial(f\varphi\psi)}{\partial(\xi\eta\zeta)}\, d\xi\, d\eta\, d\zeta = \frac{\partial(f\varphi\psi)}{\partial(\xi\eta\zeta)} \iiint_{(\omega')} d\xi\, d\eta\, d\zeta$

$$J = \lim \sum F(x_{ijk}\ y_{ijk}\ z_{ijk})\, \omega_{ijk}$$

$$= \lim \sum F(\xi_{ijk}\ \varphi_{ijk}\ \psi_{ijk}) \left(\frac{\partial(f\varphi\psi)}{\partial(\xi\eta\zeta)}\right)_{\xi_i \eta_j \zeta_k} \omega'_{ijk}$$

$$= \iiint F\left(f(\xi\eta\zeta),\ \varphi(\xi\eta\zeta),\ \psi(\xi\eta\zeta)\right) \frac{\partial(f\varphi\psi)}{\partial(\xi\eta\zeta)}\, d\xi\, d\eta\, d\zeta$$

此ハ n-ple integral ニ hold ス

Ex. $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi \qquad (r, \theta, \varphi)$

$z = r\cos\theta$

$$\frac{\partial(xyz)}{\partial(r\theta\varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_{(\Omega)} F(xyz)\, dx\, dy\, dz = \iiint_{(\Omega')} F(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\varphi$$

vol. ハ 両辺ニ F ヲ 1 トオケバヨイ

Ex. space ニ於テ三ツノ parallel planes ニテ囲マレタ parallelepiped ノ volume

$\xi \equiv \alpha_1 x + \beta_1 y + \gamma_1 z = \pm p_1$

$\eta \equiv \alpha_2 x + \beta_2 y + \gamma_2 z = \pm p_2 \qquad \alpha, \beta, \gamma$: direct. cosines

$\zeta \equiv \alpha_3 x + \beta_3 y + \gamma_3 z = \pm p_3 \qquad$ ハ plane ノ equation ト

$\xi\ \eta\ \zeta$ ニ transform

$\xi = \pm p_1$

$\eta = \pm p_2$

$\zeta = \pm p_3 \qquad$ 直角平行六面体トナル 其ノ volume

$8 p_1 p_2 p_3$

$$V = \iiint dx\, dy\, dz = \iiint \frac{\partial(xyz)}{\partial(\xi\eta\zeta)}\, d\xi\, d\eta\, d\zeta$$

$$= \iiint \frac{d\xi\, d\eta\, d\zeta}{\frac{\partial(\xi\eta\zeta)}{\partial(xyz)}} = \frac{1}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}} \underbrace{\iiint d\xi\, d\eta\, d\zeta}_{= 8p_1p_2p_3}$$

[illegible] planes [illegible] distances [illegible] constant [illegible] minimal volume parallel-piped [illegible] (— [illegible] n dimensions [illegible])

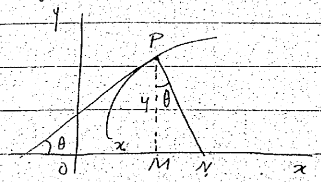
[illegible] p₁p₂p₃ const [illegible] $\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}$ [illegible] max min [illegible] Hadamard's theorem [illegible]

$$\mathrm{abs}\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix} \leqq \sqrt{(\alpha_1^2+\beta_1^2+\gamma_1^2)(\alpha_2^2+\beta_2^2+\gamma_2^2)(\alpha_3^2+\beta_3^2+\gamma_3^2)}$$

$\alpha\ \beta\ \gamma$ direct cos [illegible] $=1$ [illegible] maximum [illegible] orthogonal [illegible] max [illegible] min [illegible]

Differential Equations.

Geom. [illegible] Dynamics [illegible] simple [illegible] dif. eq. [illegible] reduce [illegible]



PN normal [illegible] constant [illegible] x axis [illegible] curve [illegible] circles [illegible] solution [illegible]

Curve [illegible] $y=f(x)$ [illegible]

$$\tan\theta = y' \qquad \sqrt{1+y'^2}=\sec\theta$$

$$PN\cos\theta = PM = y$$

$$PN = \frac{y}{\cos\theta} = y\sqrt{1+y'^2} = \text{const}$$

[illegible] satisfy [illegible] y [illegible] equation [illegible] unknown f [illegible] derived funct. [illegible] dif. eq. [illegible]

$$\left(\frac{k}{y}\right)^2 = 1+y'^2$$

$$y' = \sqrt{\left(\frac{k}{y}\right)^2-1} = \frac{dy}{dx}$$

$$\frac{dx}{dy} = \frac{y}{\sqrt{k^2-y^2}}$$

$$x = \int\frac{y\,dy}{\sqrt{k^2-y^2}} + C$$

$$x = -\sqrt{k^2-y^2} + C$$

$$(x-c)^2+y^2=k^2$$

y [illegible] define [illegible] centre [illegible] x axis [illegible] circle [illegible]

2 radius of curvature ノ constant ナル curve ヲ求ム

$$y = f(x)$$

Curvature $\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}} = \alpha$

一ノ differential eq. ナルガ又 second order ナリ

此ハ $\dfrac{y'}{(1+y'^2)^{\frac{1}{2}}}$ ヲ differentiate シタモノ

$$\frac{y'}{(1+y'^2)^{\frac{1}{2}}} = \alpha x + \beta$$

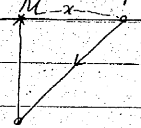
$$y' = \frac{\alpha x + \beta}{\sqrt{1-(\alpha x+\beta)^2}}$$

$$y = \int \frac{(\alpha x+\beta)\,dx}{\sqrt{1-(\alpha x+\beta)^2}} + \gamma$$

$$= \frac{1}{\alpha}\sqrt{1-(\alpha x+\beta)^2} + \gamma$$

$(\alpha y - \alpha\gamma)^2 + (\alpha x + \beta)^2 = 1$ circle

2 一ノ particle ガ一直線ノ上ヲ動キ一点ニ constraint サレ
ニ之ニ作用スル力ハ其距離ニ比例シ一点ニ向フ [illegible]



此時 x ノ方

$\dfrac{d^2x}{dt^2} = -m^2 x$ punct ナリ

dif. eq. of second order

この両辺ニ乗ズ

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -m^2\, 2x\frac{dx}{dt}$$

$$\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2\right\} = -m^2\frac{d}{dt}(x^2)$$

$$\left(\frac{dx}{dt}\right)^2 = -m^2x^2 + C$$

$$\frac{dx}{dt} = \sqrt{C - m^2x^2}$$

$$\frac{dx}{\sqrt{C-m^2x^2}} = dt$$

$$t = \int\frac{dx}{\sqrt{C-m^2x^2}} + \alpha$$

$$= \frac{1}{m}\arcsin\frac{mx}{\sqrt{C}} + \alpha$$

$$\sin(m(t-\alpha)) = \frac{mx}{\sqrt{C}}$$

$$x = \frac{\sqrt{C}}{m}\sin m(t-\alpha)$$

$\frac{2\pi}{m}$ ハ period ノ
simple harmonic moti.
ナリ

一般ニ unknown f ト其 derived f. ノ間ノ関係 equat. ヲ ordinary dif eq. ト云ヒ其ノ中ノ derived f. ノ highest order ノ order ヲ dif. eq. ノ order ト云フ

ordinary dif. eq. ハ一ノ constant ヲ含ム eq. ヨリ其 constant ヲ eliminate シテ得ラルル

$x^2+y^2+2ax+2by+c=0$ 一般ノ circle

a, b, c ノ (12) [illegible] ヲ eliminate シテ y ヲ x ノ f トシ diff. スル

$$x + yy' + a + by' = 0$$

$$y'' \left\{ \begin{array}{l} 1 + y'^2 + yy'' + by'' = 0 \\ 3y'y'' + yy''' + by''' = 0 \end{array} \right.$$

ニテ b ヲ eliminate シ

$$y'''(1 + y'^2 + yy'') = y''(3y'y'' + yy''')$$

$$y'''(1 + y'^2) - 3y'y''^2 = 0 \qquad \text{constant ナシ}$$

3rd order ノ dif. eq. ナリ

[illegible] circle ガ一般ニ満タス eq. ナリ 即 circle ノ dif. eq. ナリ 逆ニ此 dif. eq. ヲ integrate スレバ circle ノ eq. ヲ得ル

$$\frac{y'''}{y''} - \frac{3y'y''}{1+y'^2} = 0$$

$$\frac{d}{dx}\left(\log y''\right) - \frac{3}{2}\frac{d}{dx}\left(\log(1+y'^2)\right) = 0$$

$$\log y'' - \frac{3}{2}\log(1+y'^2) = c$$

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = e^c = \text{const.} \qquad \text{constant curvature ナル}$$

[illegible]

circle ノ eq. ニ 適ス

一般ノ conic section ノ dif. eq. ヲ 求メン

$$y = ax + b \pm \sqrt{lx^2 + 2mx + n}$$

$$y' = a \pm \frac{lx + m}{\sqrt{}}$$

$$y'' = \pm \frac{ln - m^2}{(lx^2 + mx + n)^{\frac{3}{2}}}$$

$$y''^{\frac{2}{3}} = \frac{(ln - m^2)^{\frac{2}{3}}}{lx^2 + mx + n}$$

$$y''^{-\frac{2}{3}} = \frac{lx^2 + mx + n}{(ln - m^2)^{\frac{2}{3}}}$$

$$\frac{d^2}{dx^2}\left(y''^{-\frac{2}{3}}\right) = \frac{2l}{(ln - m^2)^{\frac{2}{3}}}$$

$$\frac{d^3}{dx^3}\left(y''^{-\frac{2}{3}}\right) = 0 \qquad \text{diff. eq. of conic}$$

5th order ナリ

parabola ハ $\quad a^2 - (a \cdots l) = 0 \quad l = 0$

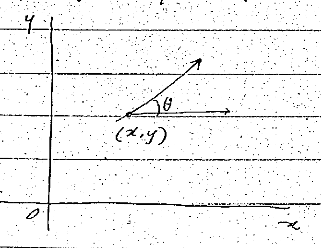
即 $\quad \frac{d^2}{dx^2}\left(y''^{-\frac{2}{3}}\right) = 0 \qquad$ 4th order

Differential equation of the first order

$$f(x, y, y') = 0$$

$$y' = \varphi(x, y) \qquad \varphi \text{ ノ one valued トス}$$

即 x, y ノ一対ニ対シ y' ハ一ツ定マル



x, y ヲ cartesian rectangular coord. ノ one point トス

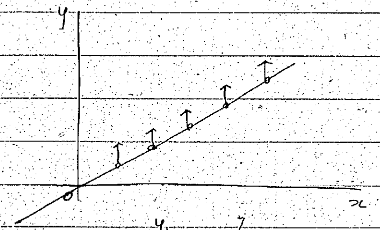
y' ハ其ノ direction ナリ

$$y' = \tan\theta$$

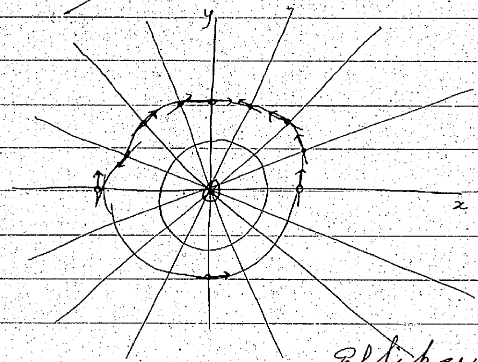
any pt が [illegible] (x, y, y') [illegible]
line element [illegible]

~~dif. eq.~~ (x, y) [illegible] line element が [illegible] ∞^2 [illegible] 此 ∞^2 line elements の defin [illegible] dif. eq. [illegible]

$y' = \frac{-K^2 x}{y}$ [illegible]



$y = mx$ [illegible]
line element [illegible]
[illegible]
$y' = -\frac{k^2}{m}$
parallel



$y = 0 \quad y' = \infty$

$x = 0 \quad y' = 0$

$y = x \quad y' = -k^2 \; (k^2 < 1)$

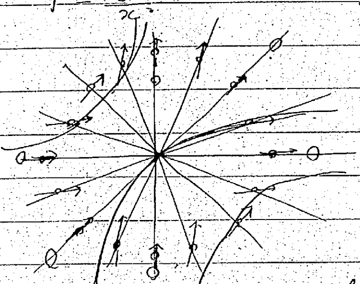
$y = -x \quad y' = k^2$

$y\,dy = -k^2 x\,dx$

$y^2 = -k^2 x^2 + C$

Ellipse $\quad y^2 + k^2 x^2 = C$

$y' = \frac{y^2}{x^2}$



$y = mx$ [illegible] $y' = m^2$

$y' = \frac{y^2}{x^2}$

$\frac{dy}{y^2} = \frac{dx}{x^2}$

$\frac{1}{y} = \frac{1}{x} + c$

$cxy + y - x = 0$

$C = 0 \quad y = x$ [illegible]

$C \neq 0$ Hyperbola $(x + \frac{1}{c})(y - \frac{1}{c}) + \frac{1}{c^2} = 0$

$y' = \varphi(x, y)$ の solution [illegible]

$y = f(x)$ [illegible]

$\left(\begin{array}{l} y' = f'(x) \\ y = f(x) \end{array}\right.$ [illegible] eq. [illegible] satisfy [illegible]

y' [illegible] $f(x)$ の tangent

$y = f(x)$ の curve の tangent の line element [illegible] line elements [illegible] curve [illegible] line elements [illegible] curve [illegible] ∞^2 [illegible] ∞^2 [illegible] line element [illegible] curve [illegible]

(x_0, y_0) [illegible] point [illegible] line element [illegible] line element [illegible] broken line [illegible] limit [illegible] curve [illegible] curve の tangent [illegible] diff. eq. [illegible] defin [illegible] line element [illegible] curve [illegible] integral curve トイフ

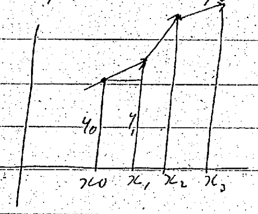
此ノ diff eq. ノ solve スルコトハ integral curve ヲ求メルコトデアル

physical ニ云ヘバ fluid ノ every particle ノ運動ガ time ニ independent ナ stationary (steady) motion ヲスルトキ $y' = f(x, y)$ ハ velocity ヲ示シテヰルトスレバ stream line ガ integral curve トナル。又 field ニ於テハ Kraftlinie トナル

此ノ curve ノ approximation ノ方法ハ数々ヲ計算スルモノデ mechanical ニ dif. eq. ノ approximate ナ solution ヲ得ル

$$y' = \varphi(x, y)$$

$(x_0 y_0)$ $\qquad y_0' = \varphi(x_0 y_0)$



$$x_1 = x_0 + \Delta x_0$$
$$y_1 = y_0 + y_0' \Delta x$$
$$x_2 = x_1 + \Delta x$$
$$y_2 = y_1 + y_1' \Delta x_1$$

一応近似ノ方法デ Graphical solution デアル

之ハ integral ヲ reduce スル。quadrature ニ帰着スル

point ノ $x\, y$ ノ ∞^2 アル integral curve ノ上

此ノ integral curve ハ arbitrary const. C ヲ含ム。∞^1 single infinity ノ integral curve ヲ

$y = F(x, C)$ ガ solution デアル

$x = x_0$, $y = y_0$ ヲ通ル curve ヲ求メルニハ initial condition トシテ $y_0 = F(x_0, C)$ カラ C ヲ決メレバヨイ

diff. eq. ノ solution ヲ finite no. of quadratures (integration) ニ reduce スルコト。$\int f(x)\, dx$ ヲ求メルコト。integral calculus ノ問題

Dif. eq. ノ場合 separate special case ヲ述ベル

1) Integration by the separation of variables

$$\frac{dy}{dx} = f(x, y) \qquad dy - f(x, y)\, dx = 0$$

$$X(x)\, dx + Y(y)\, dy = 0 \quad \text{ナル形ニ reduce サレル場合}$$

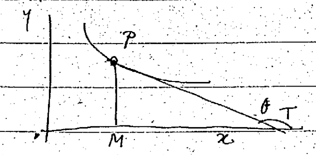
$$\int X(x)\, dx + \int Y(y)\, dy = C \quad \text{solution}$$

例 $\dfrac{dy}{dx} = f(y)$ ナドハ special case デアル

$$\frac{dy}{f(y)} - dx = 0$$

$$\int \frac{dy}{f(y)} = x + C$$

Ex. 1. tangent PT = const. ナル curve ヲ求メヨ



$$\tan\theta = \frac{dy}{dx} = y'$$

$$PT \sin\theta = y$$

$$PT = \frac{y}{\cancel{y/\sqrt{1+y'^2}}} \qquad \frac{y\sqrt{1+y'^2}}{y'}$$

$y\sqrt{1+y'^2} = ky'$ [illegible] x を explicit に [illegible]

$y' = \frac{y}{\sqrt{k^2-y^2}}$

$\frac{\sqrt{k^2-y^2}}{y}\,dy = dx$

$\int \frac{\sqrt{k^2-y^2}}{y}\,dy = x + C$

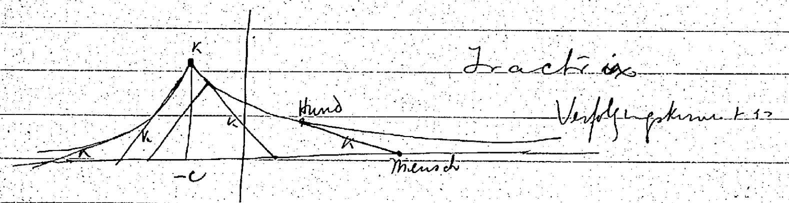
$x + C = \sqrt{k^2-y^2} - \frac{k}{2}\log\frac{k+\sqrt{k^2-y^2}}{k-\sqrt{k^2-y^2}}$

[illegible] $\begin{cases} y = k\sin\theta \\ x + C = k\cos\theta + k\log\tan\frac{\theta}{2} \end{cases}$ θ, parameter

$\theta = 0$ $y = 0$ $x \to \infty$ $(k>0)$

$\theta = \frac{\pi}{2}$ $y = k$ $x = -C$

$\theta = \pi$ $y = 0$ $x \to +\infty$



Normal, subtangent, subnormal [illegible] y と y' [illegible] func. [illegible] solve [illegible] (Constant subnormal [illegible] 1883) subtangent, subnormal [illegible] const. [illegible] Liebig [illegible]

Calculus [illegible]

Ex. 2 $y' = f(ax+by)$ $(+C)$

$ax + by = z$

$y = \frac{z-ax}{b}$

$y' = \frac{z'-a}{b}$

$\frac{z'-a}{b} = f(z)$

Ex. 3 $(x+y)\,dx - (x-y)\,dy = 0$

$(x\,dx + y\,dy) + (y\,dx - x\,dy) = 0$

$\frac{1}{2}d(x^2+y^2) + y^2 d\left(\frac{x}{y}\right) = 0$

$x^2+y^2 = u$ $y^2 = \frac{u}{1+v^2}$

$\frac{x}{y} = v$

$du + \frac{2u}{1+v^2}\,dv = 0$

$\frac{du}{u} + \frac{2\,dv}{1+v^2} = 0$

$\log u + 2\arctan v = C$

$x = r\cos\theta$ $u = r^2$

$y = r\sin\theta$ $\arctan v = \theta$

$2\log r + 2\theta = C$

$r = e^{\frac{C}{2}-\theta} = c' e^{-\theta}$ logarithmic spiral [illegible] separation [illegible]

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

$$\frac{dy}{\sqrt{1-y^2}}+\frac{dx}{\sqrt{1-x^2}}=0$$

$$\arcsin y+\arcsin x=C \qquad (1)$$

[illegible]

$$\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$$

$$y\sqrt{1-x^2}-\int\frac{xy\,dx}{\sqrt{1-x^2}} \quad \text{[illegible]} \quad xy\left(\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}\right)=0$$

$$+x\sqrt{1-y^2}-\int\frac{xy\,dy}{\sqrt{1-x^2}}=C$$

$$y\sqrt{1-x^2}+x\sqrt{1-y^2}=C \qquad (2)$$

[illegible] (1) [illegible]

(1) [illegible] (2) [illegible]

2. Homogeneous differential equations

$$P(x,y)\,dx+Q(x,y)\,dy=0$$

(m th order)

$P(x,y)$, $Q(x,y)$ same order homogeneous f

$$P(x,y)=x^mP\left(1,\frac{y}{x}\right)$$

$$Q(x,y)=x^mQ\left(1,\frac{y}{x}\right) \qquad \frac{y}{x},\ t=\frac{1}{x} \qquad P(xt,yt)=t^mP(x,y)$$

$$\frac{dy}{dx}=f\left(\frac{y}{x}\right)$$ [illegible] integral curves [illegible] similar

[illegible] homog. eq. [illegible]

$$\frac{y}{x}=u. \qquad y=xu. \qquad \frac{dy}{dx}=u+x\frac{du}{dx}$$

$$u+x\frac{du}{dx}=f(u)$$

$$\frac{du}{f(u)-x}=\frac{dx}{x}$$ separated variables

by extension

$$\frac{dy}{dx}=f\left(\frac{ax+by+c}{a'x+b'y+c'}\right)$$ [illegible] reduction [illegible]

$$x+\alpha=u \qquad ax+by+c=au+bv-a\alpha-b\beta+c$$

$$y+\beta=v \qquad a'x+b'y+c'=a'u+b'v-a'\alpha-b'\beta+c'$$

$$a\alpha+b\beta=c$$

$$a'\alpha+b'\beta=c'$$ [illegible] α,β [illegible]

$ab'-a'b\neq0$ [illegible] α,β [illegible]

$$\frac{dv}{du}=f\left(\frac{au+bv}{a'u+b'v}\right)=f\left(\frac{a+b\frac{v}{u}}{a'+b'\frac{v}{u}}\right)=F\left(\frac{v}{u}\right)$$

[illegible] homogeneous eq. [illegible] reduce [illegible]

$ab'-a'b=0$ [illegible] α,β [illegible]

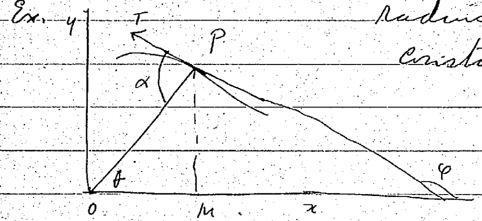
[illegible] $\frac{a}{a'}=\frac{b}{b'}=k$

$$\frac{dy}{dx}=f\left(k+\frac{h}{a'x+b'y+c'}\right)=F(a'x+b'y)$$

[illegible] separation of variables [illegible]

Ex.

Ex. radius vector と tangent と constant angle をなす curve



$$\alpha = \varphi - \theta$$
$$\tan\theta = \frac{y}{x}$$
$$\tan\varphi = y'$$

$$\tan\alpha = \frac{\tan\varphi - \tan\theta}{1 + \tan\varphi \cdot \tan\theta} = k$$

$$\frac{-\frac{y}{x} + y'}{1 - y'\frac{y}{x}} = k \qquad \text{homogeneous eq.}$$

$$y' = \frac{\frac{y}{x} + k}{1 - k\frac{y}{x}}$$

$$\frac{y}{x} = u \qquad x\frac{du}{dx} = \frac{u+k}{1-ku} - u = \frac{k(1+u^2)}{1-ku}$$

$$\frac{(1-ku)\,du}{k(1+u^2)} = \frac{dx}{x}$$

$$\frac{1}{k}\frac{du}{1+u^2} - \frac{1}{2}\frac{2u\,du}{1+u^2} = \frac{dx}{x}$$

$$\frac{1}{k}\arctan u - \frac{1}{2}\log(1+u^2) = \log x + c$$

$$\frac{1}{k}\arctan\frac{y}{x} - \frac{1}{2}\log(x^2+y^2) = c$$

$x = r\cos\theta$

$y = r\sin\theta$ $\qquad \frac{\theta}{k} - \log r = c$

$$r = c' e^{\theta/k} \qquad \text{logarithmic spiral}$$

$\alpha = \frac{\pi}{2}$ なら circle

この eq. は $(x - ky)\,dy - (y + kx)\,dx = 0$

$$(x\,dy - y\,dx) - k(y\,dy + x\,dx) = 0$$

[illegible] separate [illegible] Ex. 3 [illegible] $(k = -1)$

3. Linear differential equation

$$\frac{dy}{dx} + p(x)\,y + q(x) = 0 \qquad y,\ y' \text{ について一次である}$$

$e^{\int p(x)dx}$ を differentiate すると $e^{\int p(x)dx}\,p(x)$ となることに注意

(1)の両辺に $e^{\int p(x)dx}$ をかけると

$$\frac{d}{dx}\left(e^{\int p\,dx} \cdot y\right) + e^{\int p\,dx} \cdot q(x) = 0$$

これを integrate して

$$e^{\int p\,dx}\,y + \int e^{\int p\,dx}\,q\,dx = C$$

$$y = e^{-\int p\,dx}\left(C - \int e^{\int p\,dx}\,dx\right) \qquad \text{solution}$$

Linear eq. の解は arbitrary const C について linear である

$$y = \varphi(x) + C\,\psi(x)$$

逆に これから linear eq. [illegible] C を eliminate すれば linear eq. となる。

$y = \varphi(x) + C\,\psi(x)$

$y' = \varphi'(x) + C\,\psi'(x)$) ニテ C ヲ eliminate シ

此ノ式ハ importance ヲ有スル solution ノ中 ... particular ナルモノ ... C ...

$y_1 = \varphi(x) + C_1\psi(x)$

$y_2 = \varphi(x) + C_2\psi(x)$

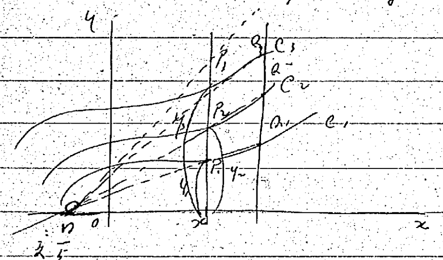
$y_3 = \varphi(x) + C_3\psi(x)$

ニテ φ, ψ ヲ eliminate シ

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{C_3 - C_1}{C_2 - C_1}$$

$$y_3 = y_1 + (y_2 - y_1)\frac{C_3 - C_1}{C_2 - C_1}$$

一般ニ $y = Ay_1 + By_2$ ニテ ... A, B ハ 一定 ...



$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{P_1P_3}{P_1P_2} = \text{Const.}$$

x = independent

$$= \frac{Q_1Q_3}{Q_1Q_2}$$

integral curves ヲ ... ordinate ニテ ... such ... 考ヘ

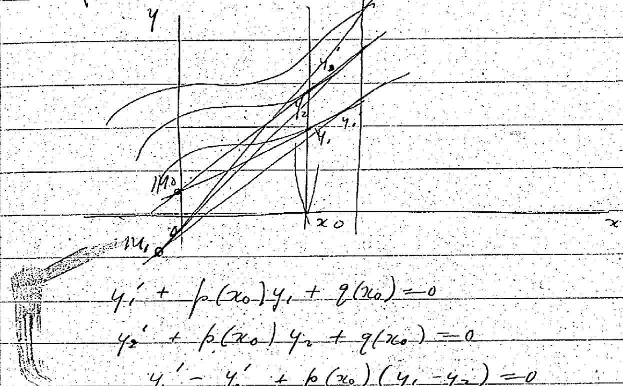
$P_1 Q_1$ ヲ ... limit ニ P_1Q_1 ... tangent

... ordinate ニテ integral curves ...

第五 ... tangent ... 一点ニ交ハ

solution ヲ graphically ... 得ル

... ordinate ニテ ... tangent ... Czuber ...



$y_1' + p(x_0)y_1 + q(x_0) = 0$

$y_2' + p(x_0)y_2 + q(x_0) = 0$

$y_1' - y_2' + p(x_0)(y_1 - y_2) = 0$

$$p(x_0) = -\frac{y_1' - y_2'}{y_1 - y_2}$$

$$q(x_0) = \frac{y_1'y_2 - y_1y_2'}{y_1 - y_2}$$

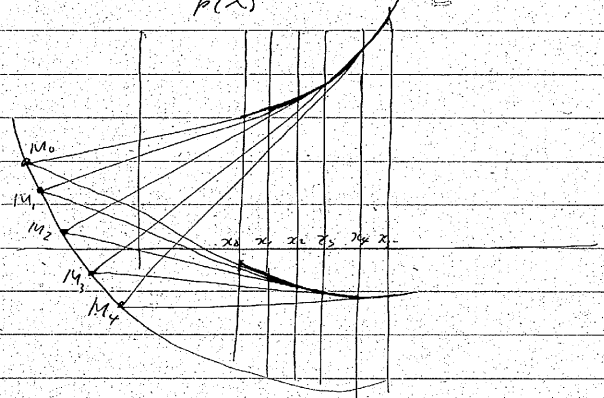
... 方程式 $\frac{\eta - y_1}{\xi - x_0} = y_1'$ $\quad \frac{\eta - y_2}{\xi - x_0} = y_2'$

交点 $\xi - x_0 = -\frac{y_1 - y_2}{y_1' - y_2'} = \frac{1}{p(x_0)}$

$$\eta = \frac{y_1'y_2 - y_1y_2'}{y_1' - y_2'} = \frac{q(x_0)}{p(x_0)}$$

x_1, x_2 ... ニテ M_1, M_2, ... 此 M ノ locus ノ eq

$$\left.\begin{aligned}\xi &= \lambda + \frac{1}{p(\lambda)} \\ \eta &= -\frac{q(\lambda)}{p(\lambda)}\end{aligned}\right\}$$ λ parameter トスル Curve ナリ



linear ニ帰スル モノ ヲ 考ヘル

4. Bernoulli's equation

$$y' = p(x)\,y + q(x)\,y^n \qquad n=0 \text{ ナルトキハ linear ナリ}$$

$$y = z^{-\frac{1}{n-1}} \qquad n \neq 1$$

$$-(n-1)\,y^{-n}\,y' = z' \quad , \quad -\frac{z'}{n-1} = p(x)\,z + q(x) \quad \text{ハ linear ナリ}$$

$$y^{-n}y' = p\,y^{-(n-1)} + q$$

Ex. 1. $\quad xy' - y + f(xy)\,y' + \varphi(xy) = 0$

$f(xy)$, $\varphi(xy)$ homogeneous , m^{th}

$\frac{y}{x} = z \qquad xy' - y + x^m f(1, \frac{y}{x}) + x^m \varphi(1, \frac{y}{x}) = 0$

$y' = z + xz' \qquad x(z + xz') - xz + x^m f(1, z)(z + xz') + x^m \varphi(1, z) = 0$

$$z'\{x^2 + x^{m+1} f(1, z)\} + x^m\{f(1, z)\,z + \varphi(1, z)\} = 0$$

$$z'\left(1 + x^{m-1} F(z)\right) + x^{m-2}\,\Phi(z) = 0$$

$$\frac{dz}{dx} \qquad \frac{dx}{dz} + \frac{F(z)}{\Phi(z)}\,x + \frac{1}{\Phi(z)}\,x^{-m+2} = 0 \qquad \text{ハ Bernoulli's eq. ナリ}$$

z ヲ indep. v. トシ $x = u^{\frac{1}{m-1}}$ トオク

Ex. 2. $\quad \{P(xy) + Q(xy)\,x\}\,y' = R(xy) + Q(xy)\,y$

P, R mth homog.

Q nth homog.

$$\left\{x^m P(1, \tfrac{y}{x}) + x^{n+1} Q(1, \tfrac{y}{x})\right\} y' = x^m R(1, \tfrac{y}{x}) + x^m Q(1, \tfrac{y}{x})$$

$\frac{y}{x} = z$

$$\{x^m P(1, z) + x^{n+1} Q(1, z)\}(z + xz') = x^m R(1, z) + x^{n+1} Q(1, z)\,z$$

$$z'\{x^{m+1} P + x^{n+2} Q\} = -(x^m P + x^{n+1})z + x^m R + x^{n+1} Q z$$

$$= x^m (R - Pz)$$

$$\frac{dx}{dz} = \frac{P}{R - Pz}\,x + \frac{Q}{R - Pz}\,x^{n+2-m} \qquad \text{Bernoulli's eq.}$$

5. Clairaut's differential equations

$$y = px + f(p) \qquad p = \frac{dy}{dx}$$

両辺ヲ x ニツキ微分スル

$$p = p + \frac{dp}{dx}\,x + f'(p)\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(f'(p) + x\right) = 0$$

1) $\frac{dp}{dx} = 0$ $\qquad$ $p = C$ $\quad$ [illegible] sol. [illegible] general solut. [illegible] $y = cx + f(c)$

2) $x + f'(p) = 0$ $\qquad$ $y = px + f(p)$

p [illegible] x, f [illegible] arbitrary Const [illegible]

[illegible] p [illegible] eliminate [illegible]

$p = p(x)$ $\qquad$ $y = p(x) \cdot x + f(p(x))$ solut.

$\therefore \frac{dy}{dx} = p(x) + \underbrace{x\frac{dp}{dx} + f'(p)\frac{dp}{dx}}_{0}$

$\therefore p(x) = \frac{dy}{dx}$

$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$ $\quad$ given eq. [illegible]

[illegible] solut. [illegible] arbitrary sol. [illegible]

[illegible] $c =$ [illegible] particular sol. [illegible]

[illegible] singular solution [illegible]

[illegible] particular sol. [illegible]

$y = cx + f(c)$ [illegible] ∞^1 [illegible]

[illegible] envelope [illegible] $y = cx + f(c)$ [illegible] $0 = x + f'(c)$

[illegible] c [illegible] eliminate [illegible]

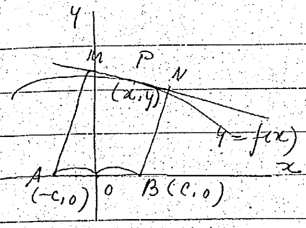
sol. [illegible] (p [illegible] eliminate [illegible]) [illegible]

singular sol. [illegible] envelop [illegible] particular sol.

[illegible] envelop [illegible] particular sol. [illegible]

[illegible]

Ex. 1



$AM \cdot BN =$ Const [illegible] curve [illegible]

tangent [illegible] eq.

$\eta - y = \frac{dy}{dx}(\xi - x)$ $\quad$ (ξ, η current coord.)

$p\xi - \eta = xp - y$

$\frac{p\xi - \eta}{\sqrt{1+p^2}} = \frac{xp - y}{\sqrt{1+p^2}}$ $\quad$ normal form $\xi\cos\theta + \eta\sin\theta = p$ dist.

AM, BN [illegible] $\frac{p\xi - \eta - xp + y}{\sqrt{1+p^2}}$ [illegible] A, B [illegible] coord. [illegible]

$\frac{pc + (y - xp)}{\sqrt{1+p^2}} = BN$, $\quad$ $\frac{-pc + (y - xp)}{\sqrt{1+p^2}} = AM$

$\frac{(y - xp)^2 - p^2c^2}{1+p^2} = a$

$y = xp \pm \sqrt{a(1+p^2) + p^2c^2}$ $\quad$ Clairaut's eq.

sol. [illegible] general. $y = kx \pm \sqrt{a + (a+c^2)k^2}$

singular, $\quad$ $0 = x \pm \frac{(a+c^2)k}{\sqrt{\ }}$, $\quad$ $y = kx \pm \sqrt{\ }$

[illegible] k [illegible] eliminate [illegible] (1) [illegible] (2) [illegible]

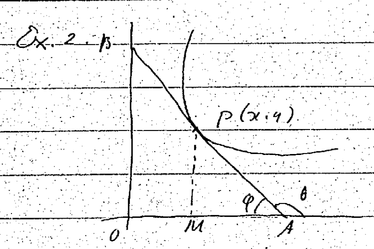
$y = \pm\frac{a}{\sqrt{\ }}$ $\qquad$ $\frac{y^2}{a} = \frac{a}{a + (a+c^2)k^2}$

$\frac{x^2}{a+c^2} = \frac{(a+c^2)k^2}{a + (a+c^2)k^2}$

$\frac{y^2}{a} + \frac{x^2}{a+c^2} = 1$

a [illegible] sign [illegible] ellipse or hyperbola, A, B foci

Ex. 2.



$AB =$ Const. な curve

$OA = x - \frac{y}{p}$

$\tan\theta = p, \quad \tan\varphi = \frac{PM}{AM}$

$AM = \frac{y}{\tan\varphi}$

$OB = -p\cdot\left(x - \frac{y}{p}\right)$

$$\left(x - \frac{y}{p}\right)^2 + \left(x - \frac{y}{p}\right)^2 p^2 = k^2$$

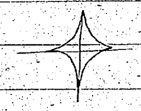
$$\frac{(y - xp)^2}{p^2} + (y - xp)^2 = k^2$$

$$y - xp = \frac{kp}{\sqrt{1+p^2}} \qquad \text{clairaut's eq}$$

general sol. $\quad y = cx + \frac{kc}{\sqrt{1+c^2}}$

$$0 = x + \frac{k}{(1+c^2)^{\frac{3}{2}}}$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}} \qquad \text{astroid}$$



此の system を sol. とする方程式は Clairaut's dif. eq. の形に書ける事が多い co' applic.

$$y = a(t)x + b(t) \qquad t \text{ parameter, co'}$$

$$\frac{dy}{dx} = p = a(t)$$

$$t = \varphi(p)$$

$$y = xp + b(\varphi(p)) \qquad \text{故に clairaut's eq}$$

となる。

5. Introduction of p as a variable

初等 extens. する

$$f(x, y, p) = 0, \qquad p = \frac{dy}{dx}$$

x につき differential し

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}p + \frac{\partial f}{\partial p}\frac{dp}{dx} = 0$$

$$\frac{dx}{dp} = -\frac{\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}}$$

$\frac{dy}{dx}$ を求むれば $\quad \frac{dy}{dp} = -\frac{p\frac{\partial f}{\partial p}}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}}$

$\frac{dx}{dp}$ = y を含まない x, p eq. となり p を variable とする

$$f(x, y, p) \equiv y - \varphi(x, p) = 0$$

$$\frac{\partial f}{\partial x} = -\frac{\partial \varphi}{\partial x}, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial p} = -\frac{\partial \varphi}{\partial p}$$

$\frac{dx}{dp} = \frac{\frac{\partial \varphi}{\partial p}}{p - \frac{\partial \varphi}{\partial x}}$ y を含まない $= \psi(x, p)$

p を indep. x を depend. v. とする dif. eq. となり solve される場合 その一般解 $x = F(p, C)$ とせよ

$$y = \varphi(x, p) = \varphi(F(p, c), p) = \Phi(p, c)$$

此の二式から p を eliminate すれば求める solution となる

又は p を parameter とすれば求める integral curve の equation となる

又

$\frac{dy}{dp}$ [illegible] y, p [illegible] eq. [illegible]

$f(x, y, p) \equiv x - \varphi(y, p) = 0$

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -\frac{\partial \varphi}{\partial y}, \quad \frac{\partial f}{\partial p} = -\frac{\partial \varphi}{\partial p}$$

$$\frac{dy}{dp} = \frac{p\frac{\partial \varphi}{\partial p}}{1 - p\frac{\partial \varphi}{\partial y}}$$ [illegible] $y = F(p, c)$ [illegible]

$$x = \varphi(y, p) = \varphi(F(p, c), p) = \Phi(p, c)$$

Ex. 1 $\quad y = \frac{a(1+p^2)^2}{p^3}$

$p = \frac{dy}{dx} \qquad dx = \frac{dy}{p}$

$$x + c = \int \frac{dy}{p} = \frac{y}{p} + \int \frac{y}{p^2} dp$$

$$= \frac{a(1+p^2)^2}{p^4} + a\int \frac{(1+p^2)^2}{p^5} dp$$

$$= \frac{a(1+p^2)^2}{p^4} + a\left\{\log p - \frac{1}{p^2} - \frac{1}{4p^4}\right\}$$

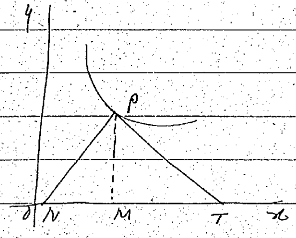
$$= a\left\{1 + \frac{1}{p^2} + \frac{3}{4p^4} + \log p\right\}$$

$$\begin{cases} x = a\left\{\frac{1}{p^2} + \frac{3}{4p^4} + \log p\right\} + b \\ y = \frac{a(1+p^2)^2}{p^3} \end{cases}$$

solution

p, any parameter

Ex. 2



$NT = 2 \cdot OM$ [illegible] curve [illegible]

subnormal + subtangent $= NT$

$$NT = \frac{y}{p} + yp = \frac{y(1+p^2)}{p}$$

$$\frac{y(1+p^2)}{p} = 2x \qquad \text{equat.}$$

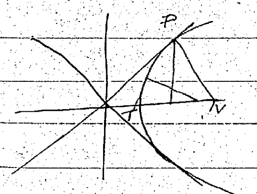
$$1 + p^2 + \frac{y(p^2-1)}{p^2}\frac{dp}{dx} = 2$$

$$\frac{y(p^2-1)}{p^2}\frac{dp}{dx} = 1 - p^2$$

$p^2 = 1$ or $\quad \frac{y}{p^2}\frac{dp}{dx} = -1$

$p = \pm 1$

$y = \pm x$



$$\frac{y}{p} = -\frac{dy}{dp}$$

$$\frac{dp}{p} = -\frac{dy}{y}$$

$$c' + \log p = -\log y$$

$$\begin{cases} y = \frac{c}{p} \\ x = \frac{c(1+p^2)}{2p^2} \end{cases}$$

$$y^2 = 2cx - c^2 = 2c\left(x - \frac{c}{2}\right)$$

parabola

envelope [illegible] $y = \pm x$ [illegible]

[illegible] homogeneous eq.

$$y' = f\left(\frac{y}{x}\right)$$ [illegible]

$$\frac{y}{x} = \varphi(p)$$ [illegible]

$$y = x\varphi(p)$$

$$p = \varphi(p) + x\varphi'(p)\frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{x\varphi'(p)}{p-\varphi(p)}$$

$$\frac{dx}{x} = \frac{\varphi'(p)}{p-\varphi(p)}dp$$

$$\log x = c + \int \frac{\varphi'(p)\,dp}{p-\varphi(p)}$$