NOTE-BOOK.

Calculus

IV

H. Tanabe

7

6. Riccati's differential equation

$$y' + ay^2 = bx^m \qquad (a, b \text{ const.})$$

此 form ニ限リ quadrature ニテ [illegible] insoluble ナル [illegible]
[illegible]

$a = 0$ or $m = 0$

$a \neq 0,\ m \neq 0 \qquad y = uz + v \qquad u, v$ funct. of x,

$\therefore\ y' = uz' + u'z + v' \qquad z$ new dep. v.

$$uz' + (u' + 2auv)z + au^2z^2 + (v' + av^2 - bx^m) = 0$$

$$u' + 2auv = 0$$

$$v' + av^2 = 0 \qquad$$ [illegible] u, v [illegible]

$$v = \frac{1}{ax} \qquad u = \frac{1}{x^2}$$

$$y = \frac{z}{x^2} + \frac{1}{ax}$$

$$z' + a\left(\frac{z}{x}\right)^2 = bx^{m+2}$$

1) $m = -2 \qquad z' + a\left(\frac{z}{x}\right)^2 = b \qquad$ homogeneous eq. integrable

2) $m \neq -2 \qquad m \neq -3$

$$x = X^{\frac{1}{m+3}}, \quad z = \frac{1}{Z}$$

$$\frac{dZ}{dX} = \frac{dZ}{dx}\frac{dx}{dX} = -\frac{dz}{dx}\,\frac{1}{m+3}\,Z^2 X^{-\frac{m+2}{m+3}}$$

$$z' = -\frac{(m+3)}{Z^2} X^{\frac{m+2}{m+3}} \frac{dZ}{dX}$$

$$a \,\Big|\, \left(\frac{z}{x}\right)^2 = \frac{1}{Z^2} X^{-\frac{2}{m+3}}$$

$$= b \,\Big|\, x^{m+2} = X^{\frac{m+2}{m+3}}$$

$$-(m+3)\frac{dZ}{dX} + aX^{-\frac{m+4}{m+3}} = bZ^2$$

$$\frac{dZ}{dX} + \frac{b}{m+3} Z^2 = \frac{a}{m+3} X$$

$y_1' + a_1 y_1^2 = b_1 x^{m_1}$ $\quad m_1 = -\frac{m+4}{m+3}$

$y' + a y^2 = b x^m$ [illegible] const [illegible]

$m_1 = 0$, or $m_1 = -2$ integrable.

or $m = -4$ or $m = -2$

故に [illegible]

$m = 0,\ m = -2,\ m = -4$

[illegible] $(m_1 \neq -2,\ m_1 \neq -3)$

$y_2' + a_2 y_2^2 = b_2 x^{m_2}$ $\quad m_2 = -\frac{m_1+4}{m_1+3}$

- - - -

$y_k' + a_k y_k^2 = b_k x^{m_k}$ $\quad m_k = -\frac{m_{k-1}+4}{m_{k-1}+3}$

$$m_k = -\frac{(2k-1)m+4k}{km+(2k+1)}$$

induction [illegible]

$$m_{k+1} = -\frac{m_k+4}{m_k+3} = -\frac{(2k+1)m+4(k+1)}{(k+1)m+(2k+3)}$$

k [illegible] hold [illegible] k+1 [illegible] holds

$$m_1 = -\frac{m+4}{m+3}$$

故に一般に [illegible]

$m_k = 0,\quad m = -\frac{4k}{2k-1}\quad k = 0, 1, 2, \dots$ [illegible]

integrable

[illegible] $k = -1, -2, -3 \dots$ [illegible] integrable

一般に $a=0,\ b=0,\ m = -\frac{4k}{2k-1},\ k = 0, \pm 1, \pm 2 \dots$

[illegible] integrable [illegible] $m = -2$ [illegible] integrable

[illegible] integrable [illegible] $m_k = -3$ [illegible]

even odd

[illegible] $4k(h+1) = (2k+1)(2h+3)$ [illegible] m_k [illegible]

[illegible] $\left(m \neq -\frac{2h+3}{h+1}\quad h < k\right)$

[illegible] Riccati [illegible] Riccati's dif eq.

[illegible] (general Riccati's dif eq.)

$$y' = p(x) y^2 + q(x) y + r(x)$$

[illegible] y [illegible] y_1 [illegible] x [illegible] funct [illegible]

integrat [illegible]

[illegible] eq. [illegible] particular sol. [illegible]

$y_0(x)$ [illegible]

$$z = y - y_0$$

$$z' + y' = p(x)(z + y_0)^2 + q(x)(z + y_0) + r(x)$$

$$y_0'(x) = p(x) y_0^2 + q(x) y_0 + r(x) \quad \text{--- --- (1)}$$

$$z' + y' = p(x)(z^2 + 2y_0 z + y_0^2) + q(x)(z + y_0) + r(x)$$

(1) [illegible] $\quad z' = p(x) z^2 + (2y_0 p(x) + q(x)) z$

y [illegible] term [illegible] Bernoulli's

eq. [illegible] solu [illegible]

$z = \frac{1}{u} \qquad z' = \frac{u'}{u^2}$

$$-\frac{u'}{u^2} = \frac{p(x)}{u^2} + (\quad)\frac{1}{u}$$

$$u' = -p(x) - (2y_0 p(x) + q(x))u$$ linear eq. ト integrable

故ニ Riccati's eq. ノ一ノ particular sol. ヲ知レバ一般ニ解ケル

之ヲ用ヰテ此ノ eq. ノ一ノ変換 [illegible] linear eq. ノ sol. ハ arbitrary const. ニ linear ニ含マレル 故ニ $u = C\varphi(x) + \psi(x)$ ト書ケル

$$y = y_0 + z = y_0(x) + \frac{1}{C\varphi(x) + \psi(x)}$$

$$= \frac{C\alpha(x) + \beta(x)}{C\gamma(x) + \delta(x)}$$

Riccati ノ eq. ノ sol. ハ linear fract. of C デアル

故ニ Riccati's eq. ハ linear eq. ニ対スル一ツノ order ノ extension ナリ

逆ニ arbitrary const. ノ linear fract. デアル fract. ハ C ヲ eliminate スレバ Riccati's eq. トナル 何トナレバ

$$y' = \frac{C^2(\gamma\alpha' + \gamma'\alpha) + C(\ ) + (\ )}{(C\gamma(x) + \delta(x))^2}$$

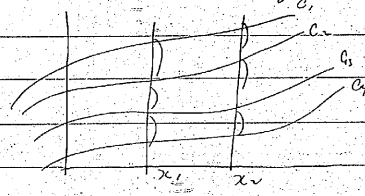
$y = $ ―― ヲ C ニ付イテ y' ヲ作ルト Riccati eq. ノ form トナル

$$y' = \frac{(\gamma\alpha' + \gamma'\alpha)(\beta - \delta y)^2 + (\ )(\beta - \delta y)(\gamma y - \alpha) + (\ )(\gamma y - \alpha)^2}{(\beta\gamma - \alpha\delta)^2}$$

故ニ $y' = p(x)y^2 + q(x)y + r(x)$

此ノ性質ハ幾何学的ニ之ヲ表ハスト次ノ如シ

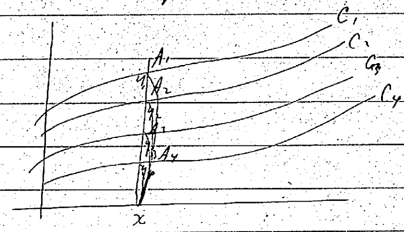
linear eq. デハ



$y = C\varphi(x) + \psi(x)$

ordinate ノ比ハ [illegible]

$$y = \frac{C\alpha(x) + \beta(x)}{C\gamma(x) + \delta(x)}$$



$$y_1 = \frac{C_1\alpha + \beta}{C_1\gamma + \delta}$$

$$y_2 = \frac{C_2\alpha + \beta}{C_2\gamma + \delta}$$

$$y_3 = \frac{C_3\alpha + \beta}{C_3\gamma + \delta}$$

$$y_4 = \frac{C_4\alpha + \beta}{C_4\gamma + \delta}$$

$$y_1 - y_3 = A_1A_3 = \frac{(\alpha\delta - \beta\gamma)(C_1 - C_3)}{(C_1\gamma + \delta)(C_3\gamma + \delta)}$$

$$y_2 - y_3 = A_2A_3 = \frac{(\alpha\delta - \beta\gamma)(C_2 - C_3)}{(C_2\gamma + \delta)(C_3\gamma + \delta)}$$

$$\frac{A_1A_3}{A_2A_3} = \frac{y_1 - y_3}{y_2 - y_3} = \frac{C_2\gamma + \delta}{C_1\gamma + \delta} \cdot \frac{C_1 - C_3}{C_2 - C_3}$$

$$\frac{A_1A_4}{A_2A_4} = \frac{y_1 - y_4}{y_2 - y_4} = \frac{C_2\gamma + \delta}{C_1\gamma + \delta} \cdot \frac{C_1 - C_4}{C_2 - C_4}$$

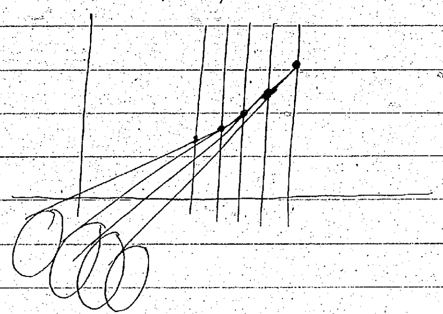
$$\frac{A_1A_3}{A_2A_3} : \frac{A_1A_4}{A_2A_4} = \frac{C_1-C_3}{C_2-C_3} : \frac{C_1-C_4}{C_2-C_4} = \frac{y_1-y_3}{y_2-y_3} : \frac{y_1-y_4}{y_2-y_4}$$

$y_1 y_2 y_3 y_4$ ノ values anharmonic ratio (double ratio)

$A_1 A_2 A_3 A_4$ ノ pt ノ anharmonic ratio

pt ノ anharmonic ratio ハ constant ノ anharmonic ratio ニ [illegible] x = [illegible] anharm. ratio [illegible]

[illegible] conic ノ envelope [illegible] tangent ハ conic ノ envelope [illegible] graphical sol. [illegible]



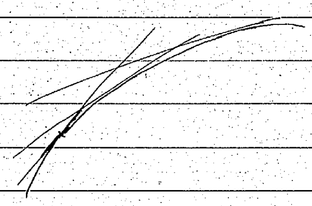
diff eq. ノ integrat [illegible] eq. ノ definition [illegible] fact ノ [illegible] Dif eq. ノ theory ノ [illegible]

Singular solution

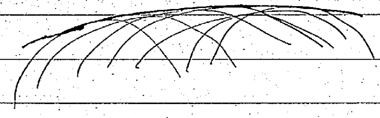
Clairaut's eq. [illegible]

$f(x, y, y') = 0$ ノ solution [illegible] eq. ヲ [illegible] ∞^2 line elements (x, y, y') [illegible] curve ノ tangent [illegible] curve ノ integral curve [illegible] solution [illegible] Clairaut's eq. [illegible] system [illegible] general sol. [illegible] line element [illegible] eq. [illegible] system of straight lines ノ envelope [illegible] touch [illegible] line element ハ curve ノ tangent [illegible] contact pt [illegible] line element [illegible] envelope ノ tangent [illegible] envelope [illegible] envelope [illegible] singular sol. [illegible] integral curve [illegible]

一般ニ



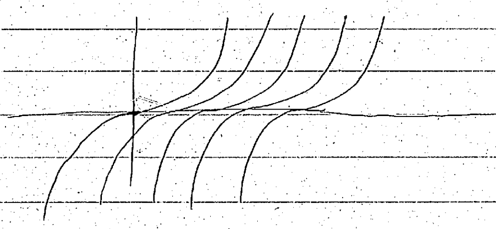
$y = \varphi(x, C)$ [illegible] ∞^1 ノ integral curves [illegible] curve ニ envelope [illegible] envelope ノ tangent [illegible] line element [illegible] eq. ヲ satisfy [illegible] envelope [illegible] singular sol. [illegible] singular sol. [illegible] 一般ニ [illegible]



$y=(x+c)^3$ ヲ solution ニモツ dif. eq. ヲ求メル

$y' = 3(x+c)^2$

$y'^3 = 27y^2$ eq.



pt. of inflexion ノ tangent ヲ求メル eq. ヲ満タス故ニ

直線 $y=0$ ハ 直線ヲ compose シテ居ル singular sol. ナリ 此場合ニ envelope トイフ言葉ヲ使用ス (Berührungskurve)

例 $y^2 = (x+c)^3$

$2yy' = 3(x+c)^2$ $\qquad y'^3 = \frac{27}{8}y$ eq.

Cusp トナル

cusp ノ locus ハ $y=0$ ナル 直線デ singular sol. ナリ

一般ニ $\begin{cases} \frac{\partial\varphi}{\partial c} = 0 \\ y = \varphi(x, C) \end{cases}$ ヨリ C ヲ eliminate シテ得タ

而シテ dif. eq. ヲ満タストキハソレヲ singular sol. トナルノデアル

更ニ sing. sol. ハ一般ニ如何ナル意味ヲ持ツカト云フニ

前ノ sol. line element ハ一ツ其ノ envelope

上ノ pt. ヲ通ル line element ニツ其ノニツ以上アル

時ハ singular sol. ヲモツトイフ ここで sing. sol. ヲ

define スル

今 $f(x, y, y') = 0$ ガ y' ノ (二次) polynomial トシ

$f(x,y,y') = p_0(x,y)y'^2 + p_1(x,y)y' + p_2(x,y) = 0$

ナル場合 y' ノ roots ガ一ノ (x,y) ニ於テ equal roots トナルトキ singular sol. ナリトス

equal roots ノ cond. ハ $\begin{cases} f(x,y,y') = 0 \\ \frac{\partial f}{\partial y'} = 0 \end{cases}$

ここで y' ヲ eliminate シテ x, y ノ f ハ一ノ Curve トナル

其ノ点ノ locus ナリ

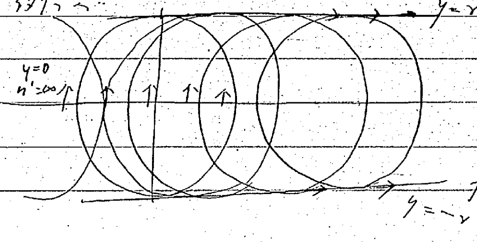
一般ニ $f=0$ $\frac{\partial f}{\partial y'} = 0$ ヲ満タス点

点ニ $f(x,y,y')=0$

$\frac{\partial f}{\partial y'} = 0$ ヲ満タスモノヲ singular line element トイフ

singular line element ヲ Curve ヲ持ツトキ其 Curve ヲ singular sol. (integral curve) トイフ

例ヘバ



$(x+c)^2 + y^2 = r^2$

$x + c + yy' = 0$

$y^2y'^2 + y^2 = r^2$

$f(x,y,y') = y^2(1+y'^2) - r^2 = 0$ eq.

$\frac{\partial f}{\partial y'} = 2y^2 y' = 0$

$y = 0$ [illegible] $y' = \infty$ sing. line element

$y' = 0$ $y = \pm x$ [illegible]

$y = 0,\ y' = \infty$ curve [illegible] circle [illegible] to [illegible]

$y' = 0$ $y = \pm x$ singular sol.

envelope [illegible] singular sol [illegible] envelope [illegible]

singular sol [illegible]

$f(x y y') = y'^2 - y^3 = 0$

$y = \frac{1}{4(x+c)^2}$

$\frac{\partial f}{\partial y'} = 2y' = 0$

$y' = 0,\ y = 0$ singular line element

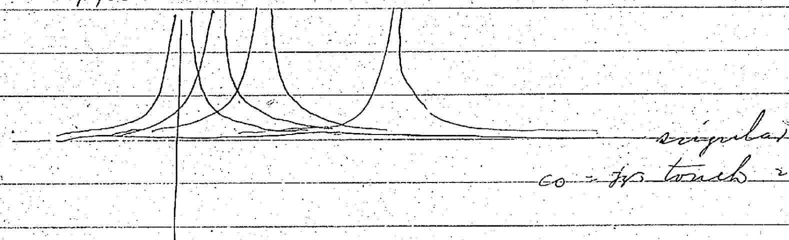
$y = 0$ singular solut.

[illegible] general sol [illegible] envelope [illegible]

particular sol [illegible] c [illegible] ∞

[illegible] $y = 0$ [illegible] limiting curve, asymtotic curve

[illegible]



singular sol.

∞ [illegible] touch [illegible]

sing sol. envelope [illegible] limiting curve [illegible]

[illegible]

singular sol [illegible] particular sol [illegible] Cases

[illegible]

Cauchy [illegible] $f(x y y') \equiv y'^3 - 4xyy' + 8y^2 = 0$

$\frac{\partial f}{\partial y'} = 3y'^2 - 4xy = 0$

$y' = \left(\frac{4}{3}xy\right)^{\frac{1}{2}}$

$\left(\frac{4}{3}xy\right)^{\frac{3}{2}} - 4xy\left(\frac{4}{3}xy\right)^{\frac{1}{2}} + 8y^2 = 0$

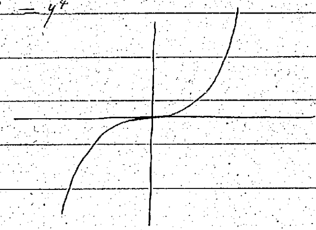
$\left\{\left(\frac{4}{3}\right)^{\frac{3}{2}} - 4\left(\frac{4}{3}\right)^{\frac{1}{2}}\right\}(xy)^{\frac{3}{2}} + 8y^2 = 0$

$-\frac{16}{3\sqrt{3}}(xy)^{\frac{3}{2}} + 8y^2 = 0$

$\frac{4}{27}(xy)^3 = y^4$

$y = 0$

$y = \frac{4}{27}x^3$



[illegible] eq. [illegible] solve [illegible]

$4x = \frac{p^3 + 8y^2}{yp}$ $y \neq 0$

$x = \varphi(y, p)$

$4 = \frac{yp(3p^2p' + 16yp) - \cdots}{y^2p^2}$

$p'(2yp^3 - 8y^3) = -4y^2p^2 + p^5$

$2yp'(p^3 - 4y^2) = p^2(p^3 - 4y^2)$

$p^3 = 4y^2$

$2yp' = p^2$ or $2y\frac{dp}{dy} = p$ $\frac{2dp}{p} = \frac{dy}{y}$ $2\log p = \log y$

$\therefore y = cp^2$

1) $p^3 = 4y^2$

$x = \frac{p^3+8y^2}{4yp}$ $\qquad x^2 = \frac{9p^6}{4p^5} = \frac{9}{4}p$

$p = \frac{4}{9}x^2$

$y^2 = \frac{p^3}{4} = \frac{1}{4}\frac{4^3}{9^3}x^6$

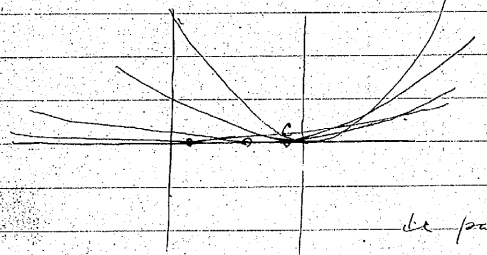
$y = \frac{4}{27}x^3$

2) $y = cp^2$

$x = \frac{1+8c^2p}{4c}$

$c\left(\frac{4cx-1}{8c^2}\right)^2 = y$

$\frac{1}{4c} = C \qquad y = C(x-C)^2$ parabola general sol.



$c=0$

$y=0$

$y=0$ singular sol.

particular sol.

parabola envelope

$y = \frac{4}{27}x^3$ curve

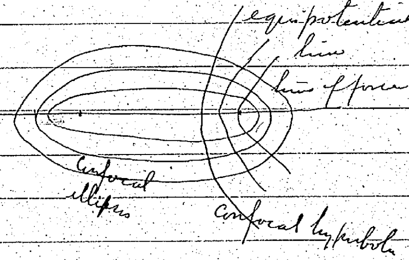
$y=0$ envelope

$y' - f(xy) = 0$ $\qquad \frac{\partial F}{\partial y'} = 1$

singular sol.

Orthogonal trajectory. $\qquad$ singular sol.

co' curves system



$y = \varphi(x, C)$

curves $\perp$

curve

system of curves

orthogonal trajectory

$y = \varphi(x, c)$ integral curves diff. eq.

$\frac{dy}{dx} = \frac{\partial \varphi}{\partial x}$

C eliminate

$f(xy\,y') = 0$

given eq. Clairaut's eq.

line element of dif. eq. satisfy $(x\,y\,y')$

$\perp$ line element $(x\,y\,\bar{y}')$

$y'\bar{y}' = -1$ $\perp$

$y' = -\frac{1}{\bar{y}'}$

$f(x, y, -\frac{1}{\bar{y}'}) = 0$

orthog. traj. line element $x\,y\,\bar{y}'$

$f(x\,y, -\frac{1}{\bar{y}'}) = 0$ satisfy

[illegible] dif. eq. [illegible] integral curves [illegible] curve [illegible]
[illegible]

$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1$ λ parameter

confocal conics

$a^2>b^2$ [illegible] $\lambda>a^2>b^2$ imaginary

$a^2\geq\lambda\geq b^2$ hyperbola

$b^2\geq\lambda$ ellipse

[illegible] satisfy [illegible] dif. eq.

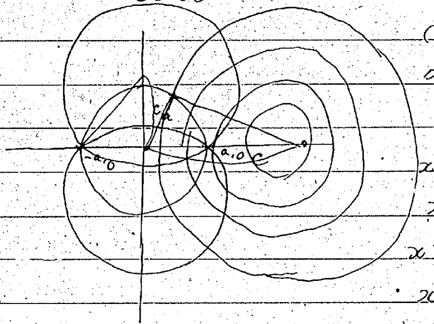
[illegible] $\frac{x}{a^2-\lambda}+\frac{yy'}{b^2-\lambda}=0$ [illegible] λ [illegible] eliminate [illegible]

$a^2-b^2=x^2-y^2-xy\left(\frac{1}{y'}-y'\right)$

orth. traj. $a^2-b^2=x^2-y^2-xy\left(-y'+\frac{1}{y'}\right)$

[illegible] eq. [illegible] curve [illegible]

[illegible] curve [illegible] orthogonal system [illegible]



$(a,0)$ $(-a,0)$ [illegible] centre

[illegible] y axis [illegible] system of

circles [illegible] orthog. traj.

$x^2+(y-c)^2=a^2+c^2$

$x+(y-c)y'=0$

$x^2+\frac{x^2}{y'^2}=a^2+\left(y+\frac{x}{y'}\right)^2$

$x^2(1+y'^2)=a^2y'^2+(x+yy')^2$

$y'^2(x^2-y^2-a^2)-2xyy'=0$

$f(x,y,y')\equiv y'(x^2-y^2-a^2)-2xy=0$

$f\left(x,y,-\frac{1}{y'}\right)\equiv-\frac{1}{y'}(x^2-y^2-a^2)-2xy=0$

$=\frac{dx}{dy}(y^2-x^2+a^2)-2xy=0$

$f(x,y,y')=\frac{dy}{dx}(x^2-y^2-a^2)-2xy=0$

[illegible] x y [illegible] a^2 [illegible]

[illegible] integral [illegible]

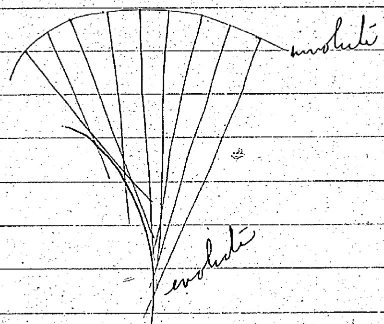
$y^2+(x-c)^2=c^2-a^2$

[illegible] orthog. traj. [illegible] eq. [illegible]

[illegible] x axis [illegible] centre [illegible] c^2-a^2 [illegible] radius [illegible] circles [illegible]

[illegible] curves [illegible] curve [illegible] tangent [illegible]

[illegible] curves [illegible] orthogonal trajectory [illegible]



[illegible] involute evolute [illegible]

[illegible] special case [illegible]

[illegible] evolute [illegible]

involute [illegible]

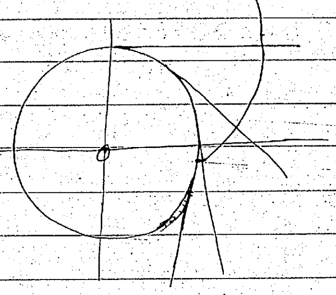
given curve $\varphi(x,y)=0$

tangents $\eta-y=y'(\xi-x)$ ξ,η current coord.

[illegible] tangent [illegible] dif. eq. [illegible] orthogonal

trajectory [illegible]

[illegible] circle [illegible] circle-involute [illegible]



origin [illegible] tangent [illegible] distance

$\eta - \xi y' = y - xy'$

$$d = \frac{y - xy'}{\sqrt{1+y'^2}}$$

circle [illegible] dist. const [illegible]

$$\frac{y - xy'}{\sqrt{1+y'^2}} = a$$ [illegible] circle [illegible] tangents [illegible] dif. eq.

orth. traj. $\quad y + \frac{x}{y'} = a\sqrt{1+\frac{1}{y'^2}}$

[illegible] dif. eq.

$$x + yy' = a\sqrt{1+y'^2}$$

$$x = -yp + a\sqrt{1+p^2} \qquad x = \varphi(y p)$$

$$1 = -p^2 - yp' + a\frac{pp'}{\sqrt{1+p^2}}$$

$$\frac{dy}{dp} + \frac{p}{1+p^2}y - \frac{ap}{(1+p^2)^{\frac{3}{2}}} = 0 \qquad \text{linear eq.}$$

$$(\sqrt{1+p^2}\, y)' = \frac{a p^2}{1+p^2}$$

$$\sqrt{1+p^2}\cdot y = c + a\int\frac{p^2\,dp}{1+p^2}$$

$$= c + a(p - \arctan p)$$

$$y = \frac{c + a(p - \arctan p)}{\sqrt{1+p^2}}$$

$$x = a\sqrt{1+p^2} - \frac{p}{\sqrt{p^2+1}}\left(c + a(p - \arctan p)\right)$$

p [illegible] parameter [illegible]

$p = \tan\theta$ [illegible]

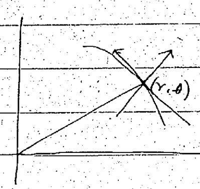
$$y = \cos\theta\,(c + a(\tan\theta - \theta))$$

$$= c\cos\theta + a\sin\theta - a\theta\cos\theta$$

$$\begin{cases} y = (c - a\theta)\cos\theta + a\sin\theta \\ x = (c - a\theta)\sin\theta + a\cos\theta \end{cases} \qquad \text{circle-involute}$$

[illegible] curve [illegible] polar coordinate [illegible]

$$f(r, \theta, r') = 0$$



$\frac{r}{r'}$ angle tangent [illegible] line element [illegible]

angle tang. $\frac{r'}{r^2}$ [illegible]

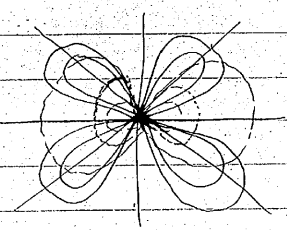
$$\frac{r}{r'} \cdot \frac{r}{\bar{r}'} = -1$$

$$-\frac{r^2}{\bar{r}'} = r'$$

$$f\left(r, \theta, -\frac{r^2}{\bar{r}'}\right) = 0$$

orth. traj. [illegible] dif. eq. $\quad f\left(r, \theta, -\frac{r^2}{\bar{r}'}\right) = 0$

Ex. $\quad r^2 = c^2\sin 2\theta \qquad$ Lemniscate



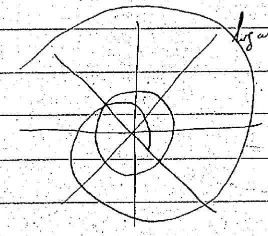
$$r' = r\cot 2\theta$$

orth. traj. $\quad r' = -r\tan 2\theta$

$$r^2 = C^2\cos 2\theta$$

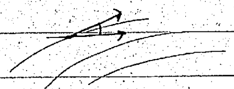
[illegible] lemniscate

orthogonal trajectory [illegible] Isogonal trajectory
[illegible] equal angle [illegible]



[illegible] curve [illegible] line element $(x\ y\ y')$
α [illegible] constant angle [illegible] line el. $(x\ y\ \bar{y}')$

$$\tan\alpha = \frac{y'-\bar{y}'}{1+y'\bar{y}'}$$



$$\bar{y}' = \frac{y'-\tan\alpha}{1+\tan\alpha\, y'}, \qquad y' = \frac{\bar{y}'+\tan\alpha}{1-\tan\alpha\, \bar{y}'}$$

given curve dif eq. $f(x\ y\ y')=0$
isogonal trajectory $f\left(x, y, \dfrac{y'+\tan\alpha}{1-\tan\alpha\, y'}\right)=0$
polar coord. [illegible]

$f(r, \theta, r')=0$ [illegible] radius vector [illegible] tangent [illegible] angle [illegible] $\frac{r}{r'}$

$$f\left(r, \theta, \frac{r(r'-\tan\alpha)}{r+\tan\alpha\, r'}\right)=0$$

[illegible]

$$\tan\alpha = \frac{\frac{r}{r'}-\frac{r}{\bar{r}'}}{1+\frac{r}{r'}\cdot\frac{r}{\bar{r}'}}$$

Some ordinary differential equations of the 2nd order.

Geom. [illegible] radius ~~vector~~ of curvature [illegible] second dif. coeff. [illegible] mechanik [illegible] acceleration [illegible] curve [illegible] motion [illegible]

1. $f(y', y'', x)=0$ [illegible] given eq. [illegible] simplest [illegible]

[illegible] y' [illegible] z [illegible] new variable [illegible]

$f(z, z', x)=0$ [illegible] 1st order [illegible] dif. eq. [illegible]

2. $f(y'', y)=0$ y', x [illegible]

[illegible] integrate [illegible]

$y'' = \varphi(y)$ [illegible]

$2y'$ [illegible] $d(y'^2) = 2\varphi(y)\,dy$

$y'^2 = 2\int\varphi(y)\,dy + c$ 1st order [illegible]

[illegible]

$y = \varphi(y'')$ [illegible]

$y''=z$ $\quad y=\varphi(z)$

$$y' = \varphi'(z)\frac{dz}{dx}$$

$$d(y'^2) = 2y'y''\,dx = 2\varphi'(z)z\,dz$$

$$y'^2 = 2\int z\,\varphi'(z)\,dz + c$$

[illegible] y', y'' [illegible] eq. [illegible] 1 [illegible] solvable

Ex. radius of curv. [illegible] (normal)^3 [illegible] proportional [illegible] curve [illegible]

$$\frac{(1+y'^2)^{\frac{3}{2}}}{y''} \quad (\text{rad. of c.}) = a\left(y\sqrt{1+y'^2}\right)^3 \ (\text{normal})$$

$$y'' = \frac{1}{ay^3}$$

$\times 2y'$

$$y'^2 = \frac{2}{a}\int \frac{dy}{y^3} + c = -\frac{1}{ay^2} + c$$

$$y' = \sqrt{\frac{acy^2-1}{ay^2}}$$

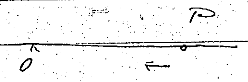
$$\frac{y\,dy}{\sqrt{\frac{acy^2-1}{a}}} = dx \qquad \frac{1}{c}\sqrt{\frac{acy^2-1}{a}} = x + c' \quad c \neq 0$$

ellipse or hyperbola

$c = 0$ [illegible] $\quad \sqrt{-a}\,\frac{y^2}{2} = x + c'$ parabola

— [illegible] : Conic [illegible]

$$y'' = \frac{1}{ay^3} \quad \frac{d^2x}{dt^2} = \frac{1}{ax^3} \ \text{[illegible]}$$



particle [illegible] O [illegible] tend [illegible] central force $\frac{1}{ax^3}$ [illegible] motion [illegible] eq. [illegible]

— [illegible] : O [illegible] dist. [illegible] function [illegible] force [illegible]

3. [illegible] $y'' \times 2y' = ((y')^2)'$ [illegible]

Ex. rad of curv $\propto$ normal

$$\frac{(1+y'^2)^{\frac{3}{2}}}{y''} = ay(1+y'^2)^{\frac{1}{2}}$$

$$\frac{1+y'^2}{y''} = ay$$

$$\frac{y''}{1+y'^2} = \frac{1}{ay}$$

$\times 2y'$ $$\frac{2y'y''}{1+y'^2} = \frac{2y'}{ay}$$

$$\log(1+y'^2) = \frac{2}{a}\log y + c'$$

$$1 + y'^2 = cy^{\frac{2}{a}}$$

[illegible] rad of curv = normal

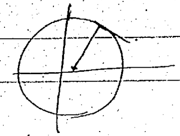
$$y' = \sqrt{\left(\frac{y}{c}\right)^2 - 1}$$

$$c\log\left(\frac{y}{c} + \sqrt{\left(\frac{y}{c}\right)^2 - 1}\right) = x + c'$$

$$\frac{y}{c} + \sqrt{} = e^{\frac{x+c'}{c}}$$

$$\frac{y}{c} - \sqrt{} = e^{-\frac{x+c'}{c}}$$

$$y = \frac{c}{2}\left(e^{\frac{x+c'}{c}} + e^{-\frac{x+c'}{c}}\right) \quad \text{Catenary}$$



circle [illegible] a [illegible]

$y' = \sqrt{\frac{c}{y^2} - 1} = \frac{\sqrt{c-y^2}}{y}$ $\qquad \frac{y\,dy}{\sqrt{c-y^2}} = dx$

convex ト concave トハ y'' ノ符号デ

second order ニ於ケル一般ニ [illegible]

Linear differential equations

(1) $y^{(n)} + p_1(x) y^{(n-1)} + p_2(x) y^{(n-2)} + \cdots + p_{n-1}(x) y' + p_n(x) y = f(x)$

於テ $y, y', \ldots y^{(n)}$ ニ linear ナル

各項ノ係数ハ [illegible]

若シ y ヲ含マヌ term $f(x) = 0$ ナラバ linear homogeneous d.e.

トイフ $f(x)$ ガ 0 ナルカナイカデ solut. ノ性質ニ

大ナル差アリ

(1)ノ particular sol. $u(x)$ ヲ知ッタ時ニハ

$u^{(n)} + p_1(x) u^{(n-1)} + \cdots + p_n(x) u = f(x)$

コレヲ (1) ヨリ引ケバ

$(y-u)^{(n)} + p_1(x)(y-u)^{(n-1)} + \cdots + p_n(x)(y-u) = 0$

即チ此ノ eq. ハ homogeneous dif. eq. デアル

故ニ一般ノ linear dif. eq. ノ particular sol. ヲ

求メルコトハ homog. linear dif. eq. ヲ解クコトニナル

(1)ノ general sol. ハ particular sol. ト homog.

eq. ノ sol. トノ和デアル

particular sol. ノ問題ハ inspection ニヨリテ知ル

[illegible] theory ハ [illegible]

例ヘバ homog. $\quad y' + y^2 = \frac{1}{x^2} \qquad y = \frac{c}{x}$

eq.(H) ノ sol. ヲ [illegible]

$y_1(x), \quad y_2(x)$

homog. ナル故 $c_1 y_1(x)$ モ sol. 又 $c_2 y_2(x)$ モ

$c_1 y_1(x) + c_2 y_2(x)$ モ sol. (linear ナル故)

一般ニ n 個ノ particular sol. ガ分レバ

$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ モ sol. [illegible]

arbitrary const. ハ n 個

逆ニ $y(x)$ [illegible] n 回 differentiate シ

c ヲ eliminate スレバ dif. eq. ヲ得ル

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

$y' = c_1 y_1' + c_2 y_2' + \cdots + c_n y_n'$

$y'' = c_1 y_1'' + c_2 y_2'' + \cdots + c_n y_n''$

$\cdots\cdots$

$y^{(n)} = c_1 y_1^{(n)} + c_2 y_2^{(n)} + \cdots + c_n y_n^{(n)}$

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ y'' & y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & & & & \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = 0$$

コレヲ expand スレバ $y\ y'\ y'' \cdots y^{(n)}$ ノ homogeneous linear eq. トナル

又 n 個ノ particular sol. ヲ[illegible] sol. ハ
$y(x)$ ノ形ヲトル [illegible]

$f_1(x), f_2(x), \ldots, f_m(x)$ ノ m 個ノ f. アリ
$c_1, c_2, \ldots, c_m$ ナル constant ニ [illegible]
[illegible]

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0 \text{ identically}$$

ナル場合ハ m 個ノ f ハ linearly dependent functions ト云フ。[illegible] $c_1, \ldots, c_m$ ガ凡テ 0 ナル場合ニ限ル [illegible] ハ linearly independent ナリトス

例ヘバ $f_1(x) = 1 \quad f_2(x) = x \quad f_3(x) = x^2, \ldots$

$f_m(x) = x^{m-1}$

ハ linearly indep. ナリ

$$c_1 + c_2 x + c_3 x^2 + \cdots + c_m x^{m-1} = 0$$

ガ identically $= 0$ ナルハ c ガ凡テ 0 ノ場合ニ限ル

$f_1(x)\ f_2(x) \ldots f_m(x)$ ガ linearly dependent ナル為ノ necessary & suff. condition ハ

$$D \equiv \begin{vmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{vmatrix} = 0 \text{ identically } [illegible]$$

Proof. linearly dep. ナリトセバ

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m = 0 \text{ ナリ}$$

之ヲ differentiate シ

$$m \left\{ \begin{array}{l} c_1 f_1' + c_2 f_2' + \cdots + c_m f_m' = 0 \\ c_1 f_1'' + c_2 f_2'' + \cdots + c_m f_m'' = 0 \\ \cdots\cdots \\ c_1 f_1^{(m-1)} + c_2 f_2^{(m-1)} + \cdots + c_m f_m^{(m-1)} = 0 \end{array} \right.$$

m 個ノ C ニ就テノ一次方程式 (homog. linear eq.) [illegible] D=0 ナラザルベカラズ
D=0 ナラザレバ 此方程式ノ [illegible] C ガ 0 ナラザルベカラズ [illegible]
故ニ D=0 ハ necessary
次ニ sufficiency ヲ証ス
D=0 ナレバ [illegible] 0 ナラズ $c_1, c_2, \ldots, c_m$ [illegible]

last line ニテ expand シ

$$D = f_1^{(m-1)} D_1 + f_2^{(m-1)} D_2 + \cdots + f_m^{(m-1)} D_m$$

$D_1, \ldots, D_m$ ガ凡テ 0 ナル場合ハ [illegible]
$f_1(x) \ldots f_{m-1}(x)$ [illegible]
Determinant ノ性質ニヨリ

$$f_1 D_1 + f_2 D_2 + \cdots + f_m D_m = 0$$

$$f_1' D_1 + f_2' D_2 + \cdots + f_m' D_m = 0$$

$m-1$ $\left\{ \begin{array}{l} - - - - - \\ f_1^{(m-2)} D_1 + f_2^{(m-2)} D_2 + \cdots + f_m^{(m-2)} D_m = 0 \\ f_1^{(m-1)} D_1 + f_2^{(m-1)} D_2 + \cdots + f_m^{(m-1)} D_m = D = 0 \text{ ([illegible])} \end{array} \right.$

differentiate with respect to x

$m-1$ $\left\{ \begin{array}{l} f_1 D_1' + f_2 D_2' + \cdots + f_m D_m' = 0 \quad (f_1' D_1 + \cdots = 0 \\ f_1' D_1' + f_2' D_2' + \cdots + f_m' D_m' = 0 \\ - - - - - \\ f_1^{(m-2)} D_1' + f_2^{(m-2)} D_2' + \cdots + f_m^{(m-2)} D_m' = 0 \end{array} \right.$

$D_1, D_2 \cdots$ m [illegible]

$D_1', D_2' \cdots$ m [illegible]

[illegible] $D_1 : D_2 : D_3 : \cdots : D_m$ [illegible]

$D_1' : D_2' : D_3' : \cdots : D_m'$ [illegible]

equal [illegible]

$$\frac{D_1'}{D_1} = \frac{D_2'}{D_2} = \cdots = \frac{D_m'}{D_m} = \lambda(x)$$

$$\log D_1 = c + \int \lambda(x)\,dx$$

$$D_1 = c_1 e^{\int \lambda(x)\,dx}$$

[illegible]

$$D_k = c_k e^{\int \lambda(x)\,dx}$$

$f_1 D_1 + \cdots$ [illegible]

$$e^{\int \lambda(x)\,dx} (c_1 f_1 + c_2 f_2 + \cdots + c_m f_m) = 0$$

[illegible] c_k [illegible] 0 [illegible] D_k [illegible] 0 [illegible]

[illegible]

[illegible] $f_k(x)$ linearly dep eq [illegible]

$D = 0$ [illegible] linearly independent [illegible]

[illegible] theorem [illegible] dif eq [illegible]

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = 0$$

$y_1(x), y_2(x), \cdots, y_n(x)$ [illegible] linearly independent particular sol [illegible]

$$\left. \begin{array}{l} y_1^{(n)} + p_1(x) y_1^{(n-1)} + \cdots + p_n(x) y_1 = 0 \\ y_2^{(n)} + p_1 y_2^{(n-1)} + \cdots + p_n y_2 = 0 \\ - - - - - - - - \\ y_n^{(n)} + p_1 y_n^{(n-1)} + \cdots + p_n y_n = 0 \end{array} \right\} n$$

[illegible] sol y [illegible]

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$$

[illegible] $p_1, p_2, \cdots p_n$ [illegible] $n+1$ [illegible] linear eq. [illegible]

[illegible] determinant [illegible] 0 [illegible]

$$\begin{vmatrix} y^{(n)} & y^{(n-1)} & \cdots & y' & y \\ y_1^{(n)} & y_1^{(n-1)} & & y_1' & y_1 \\ \vdots & \vdots & & \vdots & \vdots \\ y_n^{(n)} & y_n^{(n-1)} & & y_n' & y_n \end{vmatrix} = 0 \text{ identically}$$

此ノ $y, y_1, y_2, \ldots y_n$ — $n+1$ functions

ト nth derived f. ヲ含ム [illegible] ノ $m = n+1$ ナル case ニ於テ

若シ $y, y_1, y_2, \ldots y_n$ ガ linearly dependent トナレバ

即チ $C_0, C_1, \ldots C_n$ ヲ [illegible]

$C_0y + C_1y_1 + \cdots + C_ny_n = 0$ identically ナル [illegible]

C ハ [illegible] 0 トナラナイ

$C_0 \neq 0$ ナレバ $y, y_1 \ldots y_n$ ハ linearly dep.

[illegible]

$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ $\quad \frac{c_1}{c_0} = C_1 \ldots$

即 linearly indep. ノ n 個ノ particular sol.

[illegible] sol. ハ [illegible] 此形ヲトル

$C_1 \ldots C_n$ ヲ arbitrary constants トスレバ

general solution トナル 此形ノ f. ハ

此ノ linear homogeneous dif. eq. ノ [illegible]

$y_1, y_2, \ldots y_n$ ノ system ヲ fundamental system of homogeneous linear dif. eq.

ト云フ 此 fund. system ヲ知レバ general sol [illegible]

近世ノ Theory of dif. eq. ハ dif. eq. ヲ define スル function ノ theory ナリ 此 linear dif. eq.

ニ関スル [illegible] function ニ

(Vf. [illegible])

此 linear dif. f. [illegible]

$y = e^x \qquad y = \log x \qquad y = \sin x, \cos x$

$y' = y \qquad y' = \frac{1}{x} \qquad y'' = -y$

integration ノ repetition [illegible] y_1, y_2 [illegible]

[illegible]

$y^{(n)} + p_1y^{(n-1)} + p_2y^{(n-2)} + \cdots + p_ny = 0$

$p_1, p_2, \ldots p_n$ constants

即 linear homog. eq. with constant coefficients

$y = e^{\rho x}$ ト置ク

$y' = \rho e^{\rho x}$

$y'' = \rho^2 e^{\rho x}$

$\ldots$

$e^{\rho x}\{\rho^n + p_1\rho^{n-1} + p_2\rho^{n-2} + \cdots + p_{n-1}\rho + p_n\} = 0$

ナラバ y ハ sol. ナリ

$e^{\rho x} \neq 0$

$\varphi(\rho) = \rho^n + p_1\rho^{n-1} + \cdots + p_{n-1}\rho + p_n = 0$ nth algebraic eq.

ヲ characteristic equation ト云フ

其 roots $\rho_1, \rho_2, \ldots \rho_n$ real or imaginary

equal roots [illegible]

$y_1 = e^{\rho_1 x}, \quad y_2 = e^{\rho_2 x}, \quad \ldots \quad y_n = e^{\rho_n x}$ [illegible]

fundamental system ヲ [illegible] 即 linearly indep.

ノ particular sol. トナル

$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ general sol.

$y_1, y_2 \cdots$ indep. [illegible]

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$ [illegible]

$$= e^{(\rho_1+\rho_2+\cdots+\rho_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & & \rho_n \\ \rho_1^2 & \rho_2^2 & & \rho_n^2 \\ \vdots & \vdots & & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & & \rho_n^{n-1} \end{vmatrix}$$

$$= e^{(\rho_1+\cdots)x} (-1)^{1+2+\cdots+n} (\rho_1-\rho_2)(\rho_1-\rho_3)\cdots(\rho_1-\rho_n)$$
$$(\rho_2-\rho_3)\cdots(\rho_2-\rho_n)$$
$$(\rho_3-\rho_4)\cdots(\rho_3-\rho_n)$$
$$\cdots$$
$$(\rho_{n-1}-\rho_n)$$

ρ_i [illegible] 0 [illegible]

$\therefore y_1, y_2 \cdots y_n$ linearly indep.

imaginary roots [illegible] conjugate [illegible]

ρ [illegible] real [illegible]

$\rho_1 = \alpha + \beta i$

$\rho_2 = \alpha - \beta i$

$C_1 y_1 + C_2 y_2 = C_1 e^{(\alpha+\beta i)x} + C_2 e^{(\alpha-\beta i)x}$

$= C_1 e^{\alpha x}(\cos \beta x + i \sin \beta x)$
$\quad + C_2 e^{\alpha x}(\cos \beta x - i \sin \beta x)$
$= C_1' e^{\alpha x} \cos \beta x + C_2' e^{\alpha x} \sin \beta x$

simple harmonic mot. [illegible]

$y'' + m^2 y = 0$

$\varphi(\rho) = \rho^2 + m^2 = 0$ characteristic eq.

$\rho_1 = im$

$\rho_2 = -im$

$y = C_1 e^{imx} + C_2 e^{-imx}$

$= C_1' \cos mx + C_2' \sin mx$

periodic [illegible] $\rho^2 - m^2 = 0$ [illegible] all real roots [illegible]

periodic [illegible]

[illegible] characteristic eq. [illegible] equal roots [illegible]

[illegible]

$\varphi(\rho) = 0$ [illegible] multiple roots [illegible]

ρ_1 m-ple roots [illegible]

$\varphi(\rho) = \varphi(\rho_1 + (\rho - \rho_1))$ by Taylor's theorem

$$= \varphi(\rho_1) + \frac{\rho-\rho_1}{1!}\varphi'(\rho_1) + \frac{(\rho-\rho_1)^2}{2!}\varphi''(\rho_1) + \cdots + \frac{(\rho-\rho_1)^{m-1}}{(m-1)!}\varphi^{(m-1)}(\rho_1)$$
$$+ \frac{(\rho-\rho_1)^m}{m!}\varphi^{(m)}(\rho_1) + \cdots + \frac{(\rho-\rho_1)^n}{n!}\varphi^{(n)}(\rho_1)$$

$\varphi(\rho)$ [illegible] n th polynomial [illegible] $\varphi^{(n+1)}$ [illegible] 0

$\varphi(\rho)$ [illegible] $(\rho-\rho_1)^m$ [illegible] divisible

[illegible] $\varphi(\rho_1) = 0 \quad \varphi'(\rho_1) = 0 \cdots \varphi^{(m-1)}(\rho_1) = 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\therefore\ i\sin\theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots$$

$$i\sin\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

$$= e^{i\theta}.$$

此ノ ρ_1 ガ m-ple roots ナルタメノ necessary & suff. cond. ハ

$$\varphi^{(m)}(\rho_1) \neq 0$$

$e^{\rho_1 x}$ ガ particular sol. ナルコトハ [illegible]

$$p_n \quad y = e^{\rho_1 x}\int z\,dx \qquad z ヲ [illegible]$$

$$p_{n-1} \quad y' = \rho_1 e^{\rho_1 x}\int z\,dx + e^{\rho_1 x} z$$

$$p_{n-2} \quad y'' = \rho_1^2 e^{\rho_1 x}\int z\,dx + 2\rho_1 e^{\rho_1 x} z + e^{\rho_1 x} z'$$

$$1 \quad y^{(n)} = \rho_1^n e^{\rho_1 x}\int z\,dx + \binom{n}{1}\rho_1^{n-1} e^{\rho_1 x} z + \binom{n}{2}\rho_1^{n-2} e^{\rho_1 x} z' + \cdots + e^{\rho_1 x} z^{(n-1)}$$

$$e^{\rho_1 x}\left\{\int z\,dx\cdot\varphi(\rho_1) + z\,\varphi'(\rho_1) + z'\frac{\varphi''(\rho_1)}{2!} + z''\frac{\varphi'''(\rho_1)}{3!} + \cdots + z^{(m-2)}\frac{\varphi^{(m-1)}(\rho_1)}{(m-1)!} + \cancel{z^{(m-1)}\varphi} + \cdots + z^{(n-1)}\frac{\varphi^{(n)}(\rho_1)}{n!}\right\} = 0$$

ナルカラ z ヲ [illegible] ナル

$$\rho^{\rho_1 x}\left\{z^{(m-1)}\frac{\varphi^{(m)}(\rho_1)}{m!} + \cdots + z^{(n-1)}\frac{\varphi^{(n)}(\rho_1)}{n!}\right\} = 0$$

[illegible] linearly indep. ナル z ヲ [illegible]

$$z = C_0' + C_1'x + C_2'x^2 + \cdots + C_{m-2}'x^{m-2}$$ ナル polynomial

[illegible] satisfy ノ solution トナル

C' ハ any arbitrary const.

$$y = e^{\rho_1 x}\int z\,dx = e^{\rho_1 x}\left(C_0 + C_1 x + C_2 x^2 + \cdots + C_{m-1}x^{m-1}\right)$$

故ニ ρ_1 ガ m-ple roots ナル

$$\underbrace{e^{\rho_1 x},\ x e^{\rho_1 x},\ x^2 e^{\rho_1 x},\ \cdots x^{m-1} e^{\rho_1 x}}_{m}$$ ノ particular

sol. ヲ得 [illegible] linearly indep. [illegible] m 個 [illegible]

故ニ一般ニ [illegible] group ヲ [illegible] 合 [illegible] n 個ノ

particular solution ヲ [illegible] fundamental

system [illegible] fundamental system [illegible] Determinant

ノ [illegible]

此 constant coefficients [illegible] treat [illegible]

[illegible]

$$x^n y^{(n)} + p_1 x^{n-1} y^{(n-1)} + \cdots + p_{n-1} x y' + p_n = 0$$

$$(a+bx)^n y^{(n)} + p_1 (a+bx)^{n-1} y^{(n-1)} + \cdots + p_{n-1}(a+bx) y' + p_n = 0$$

$x = e^t$; $a + bx = e^t$ ト置ケバ constant coeff. ニ [illegible]

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

$$x y' = \frac{dy}{dt}$$

[illegible] $y = e^{\rho x}$, 代ニ $y = x^{\rho}$, $y = (ax+b)^{\rho}$

ト置キ ρ ノ [illegible] eq. [illegible] characteristic eq. [illegible]

$$y' = \rho x^{\rho-1},\quad y'' = \rho(\rho-1)x^{\rho-2},\ \cdots\ y^{(n)} = \rho(\rho-1)\cdots(\rho-n+1)x^{\rho-n}$$

$$p_n \qquad p_{n-1}x \qquad x^n$$

$$0 = x^{\rho}\left\{\rho(\rho-1)\cdots(\rho-n+1) + p_1\rho(\rho-1)\cdots(\rho-n+2) + \cdots + p_{n-2}\rho(\rho-1) + p_{n-1}\rho + p_n\right\}$$

ρ ノ nth degree ノ equation

$$y_1 = x^{\rho_1},\quad y_2 = x^{\rho_2},\ \cdots\ y_n = x^{\rho_n}$$

equal roots [illegible]

ρ_1 mpl. roots $\quad x^{\rho_1},\ x^{\rho_1}\log x,\ x^{\rho_1}(\log x)^2, \dots x^{\rho_1}(\log x)^{m-1}$

$\because\quad x = e^t \quad t = \log x \quad e^{\rho_1 t}t^k = x^{\rho_1}(\log x)^k$

Method of variation of constants (or parameters)

[illegible] second order [illegible] (linear eq. not homog.)

(1) $y'' + p(x)\,y' + q(x)\,y = f(x)$

(2) $y'' + p(x)\,y' + q(x)\,y = 0$ [illegible] homog. [illegible] solution [illegible]

$y_1,\ y_2$ fundamental system

(2) [illegible] sol. $\therefore\ y = C_1 y_1 + C_2 y_2$

C_1, C_2 [illegible] x [illegible] function [illegible] (1) [illegible] sol. [illegible]

[illegible]

$y' = C_1 y_1' + C_2 y_2' + C_1' y_1 + C_2' y_2$

$C_1' y_1 + C_2' y_2 = 0$ [illegible] $\quad$ (3)

$y'' = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2'$

(1) [illegible] $(y,\ y',\ y'')$ [illegible] y_1, y_2 [illegible] (2) [illegible] sol. [illegible]

[illegible]

$C_1' y_1' + C_2' y_2' = f(x)$ $\quad$ (4)

(3), (4) [illegible] $C_1',\ C_2'$ [illegible]

$$C_1' = \frac{-f(x)y_2}{y_1y_2' - y_1'y_2} \qquad C_2' = \frac{f(x)y_1}{y_1y_2' - y_1'y_2}$$

$y_1y_2' - y_1'y_2$ [illegible] y_1, y_2 [illegible] fundamental system [illegible] linearly indep. [illegible] identically 0 [illegible]

C_1', C_2' [illegible] integrate [illegible] $C_1,\ C_2$ [illegible]

$$C_1 = A - \int \frac{y_2 f(x)}{y_1y_2' - y_1'y_2}\,dx$$

$$C_2 = B + \int \frac{y_1 f(x)}{y_1y_2' - y_1'y_2}\,dx$$

general sol. $\quad y = Ay_1 + By_2 + y_2\int \frac{y_1 f}{y_1y_2' - y_1'y_2}\,dx - y_1\int \frac{y_2 f}{\quad''\quad}\,dx$

$$y = Ay_1(x) + By_2(x) + \varepsilon\int e^{-\int p(\xi)d\xi}\big(y_2(x)y_1(\xi) - y_1(x)y_2(\xi)\big)f(\xi)\,d\xi$$

$Ay_1 + By_2$ [illegible] (2) [illegible] general sol.

[illegible] $y_2\int\ dx$, $y_1\int\ dx$ [illegible] (1) [illegible] particular sol. [illegible]

[illegible] homogeneous eq. solve [illegible] linear eq. solve [illegible]

$y_1y_2' - y_1'y_2$ [illegible]

$$\begin{array}{c|l} y_2 & y_1'' + p(x)y_1' + q(x)y_1 = 0 \\ y_1 & y_2'' + p(x)y_2' + q(x)y_2 = 0 \end{array}$$

$(y_1''y_2 - y_2''y_1) + p(x)(y_1'y_2 - y_1y_2') = 0$

$\frac{d}{dx}\log(y_1'y_2 - y_1y_2') = -p(x)$

$\log(\qquad) = -\int p(x)\,dx + \text{const.}$

$$\frac{1}{y_1y_2' - y_1'y_2} = \varepsilon e^{-\int p\,dx}$$

[illegible] method [illegible] generalize [illegible]

Ex. $y'' + m^2 y = f(x)$ $\qquad \frac{d^2x}{dt^2} + m^2 x = f(t)$ forced vibration

$y'' + m^2 y = 0$

$\varphi(\rho) = \rho^2 + m^2 = 0$

$\rho_1 = im, \quad \rho_2 = -im$

$\bar{y}_1 = e^{\rho_1 x} = e^{imx} = \cos mx + i \sin mx$

$\bar{y}_2 = e^{\rho_2 x} = e^{-imx} = \cos mx - i \sin mx$

$y_1 = \cos mx, \quad y_2 = \sin mx$ [illegible] fund. syst. [illegible]

$y = C_1 \cos mx + C_2 \sin mx = \alpha \sin(mx + \beta)$

$y_1' = -m \sin mx \qquad y_2' = m \cos mx$

$y_1 y_2' - y_1' y_2 = m$

$y = \alpha \sin(mx + \beta) + \frac{1}{m} \int f(\xi) \sin m(x - \xi)\, d\xi$

[illegible] $y'' + (m^2 - p(x))\, y = 0$ [illegible] homog. eq. [illegible]

$y'' + m^2 y = p(x)\, y$ [illegible]

$p(x)\, y$ [illegible]

$y(x) = \alpha \sin(mx + \beta) + \frac{1}{m} \int p(\xi) \sin m(x - \xi)\, y(\xi)\, d\xi$

[illegible] sol. [illegible] integral equation

[illegible]

$\varphi(x) = F(x) + \int_0^{\xi} K(x, \xi)\, \varphi(\xi)\, d\xi$

$F(x),\ K(x, \xi)$ known function

$\varphi(x)$ unknown, sought after.

[illegible] linear integral equation [illegible] (second kind) [illegible]

$F(x) = \int_0^{\xi} k(x, \xi)\, \varphi(\xi)\, d\xi$ [illegible] first kind [illegible]) Theory of integral eq. [illegible]

Variation of const. [illegible] method [illegible] any order [illegible] linear [illegible]

$p_n(x)\quad y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \quad (+ C_1' y_1 + \cdots + C_n' y_n = 0)$

$y' = C_1 y_1' + C_2 y_2' + \cdots + C_n y_n' \quad (+ C_1' y_1' + C_2' y_2' + \cdots + C_n' y_n' = 0)$

$y'' = C_1 y_1'' + C_2 y_2'' + \cdots + C_n y_n'' \quad (+ C_1' y_1'' + \cdots + C_n' y_n'' = 0)$

$p_1(x)\quad y^{(n-1)} = C_1 y_1^{(n-1)} + C_2 y_2^{(n-1)} + \cdots + C_n y_n^{(n-1)} \quad (+ C_1' y_1^{(n-2)} + \cdots + C_n' y_n^{(n-2)} = 0)$

$1\quad y^{(n)} = C_1 y_1^{(n)} + \cdots + C_n y_n^{(n)} + C_1' y_1^{(n-1)} + \cdots + C_n' y_n^{(n-1)} = f(x)$

linear eq. [illegible]

$y'' + y'^2 f(y) + F(y) = 0$

$y'' + y'^2 f(y) = 0$ [illegible]

$\frac{y''}{y'} + y' f(y) = 0$

$d \log y' + f(y)\, dy = 0$

$\log y' + \int f(y)\, dy = \text{const.}$

$y' e^{\int f(y) dy} = \text{const.}$

$\frac{dC}{dx} = e^{\int f(y) dy} \left(y'' + f(y) y'^2 \right) = -F(y)\, e^{\int f(y) dy}$

$C \frac{dC}{dx} = -F(y)\, e^{2\int f(y) dy} \frac{dy}{dx}$

$$\frac{c^2}{2} = \text{const} - \int F(y)\, e^{2\int f(y)dy}\, dy$$

$y'' + p(x)y' + q(x)y = 0$ homog. eq. of 2nd degree

$y_1(x),\ y_2(x)$ fund. syst.

$$\frac{y_1(x)y_2'(x) - y_1'(x)y_2(x)}{y_1(x)^2} = \frac{Ce^{-\int p(x)dx}}{y_1^2(x)}$$

$$\frac{d}{dx}\left(\frac{y_2(x)}{y_1(x)}\right) = \frac{Ce^{-\int p dx}}{y_1^2(x)}$$

$$\frac{y_2(x)}{y_1(x)} = c' + c\int \frac{e^{-\int p dx}}{y_1^2(x)}\, dx$$

故ニ particular sol. [illegible]

$$y_2(x) = C'y_1(x) + Cy_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}\, dx$$

故ニ linearly independent ナ fund. syst. トシテ

$y_1(x),\ y_1(x)\int \frac{e^{-\int p dx}}{y_1^2}\, dx$ ヲトレバヨイ

一ツノ sol. ヲ inspect [illegible]

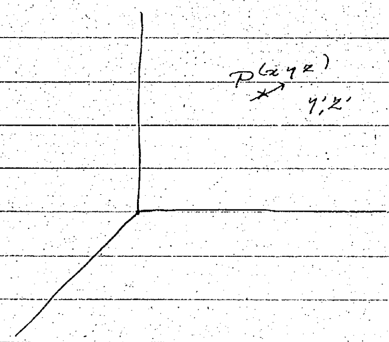
二階ノ dif. eq. [illegible] integrat. [illegible] expansion into inf. series in the sol. [illegible]

Simultaneous linear differential equati.

今 indep. x ト dep. y トノ eq. [illegible] dep. v. y, z [illegible]

$$\begin{cases} \dfrac{dy}{dx} = f(x, y, z) \\[2mm] \dfrac{dz}{dx} = \varphi(x, y, z) \end{cases}$$ 1st order 1st degree

[illegible] y, z ナル funct. [illegible] plane [illegible] line element integral curve [illegible] space ニ於ケル (x, y, z, y', z') ナル line element ハ ∞^3 (triple infinity) [illegible] space curve [illegible] ∞^1 line element [illegible] space curve [illegible] ∞^2 [illegible] 故ニ soluti. two arbitrary const. ヲ含ム



$$y = y(x, \alpha, \beta)$$
$$z = z(x, \alpha, \beta)$$

一般ニ n 個ノ dependent variable [illegible]

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots y_n)$$
$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots y_n)$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \cdots y_n)$$

ノ sol. ハ

$$y_1 = \varphi_1(x; c_1, c_2, \cdots c_n)$$
$$y_2 = \varphi_2(x; c_1, c_2 \cdots c_n)$$
$$\cdots\cdots$$
$$y_n = \varphi_n(x; c_1, c_2, \cdots c_n)$$

3 linear homog. simult. eq. ヲトル

$$\left.\begin{aligned} \frac{dy_1}{dx} &= a_1 y_1 + a_2 y_2 + a_3 y_3 \\ \frac{dy_2}{dx} &= b_1 y_1 + b_2 y_2 + b_3 y_3 \\ \frac{dy_3}{dx} &= c_1 y_1 + c_2 y_2 + c_3 y_3 \end{aligned}\right\}$$

$\bar{y}_1, \bar{y}_2, \bar{y}_3$ ヲ particular sol. トスレ

$A\bar{y}_1, A\bar{y}_2, A\bar{y}_3$ モ亦 sol. ナリ

又 $\bar{\bar{y}}_1, \bar{\bar{y}}_2, \bar{\bar{y}}_3$ ガ sol. ナラバ

$B\bar{\bar{y}}_1, B\bar{\bar{y}}_2, B\bar{\bar{y}}_3$ モ sol. ナリ

故ニ $A\bar{y}_1 + B\bar{\bar{y}}_1, A\bar{y}_2 + B\bar{\bar{y}}_2, A\bar{y}_3 + B\bar{\bar{y}}_3$ モ sol. ナリ

若シ三組ノ sol. ヲ知レバ其ノ arbitrary const. ヲ含ムカラソレモ sol. ナリ 従テ const. ヲ含ムノデ general sol. トナル

(a)(b)(c) ガ const. ナラバ soluble デアル故 import. ナリ

此ノ syst. ヲ一般ニ d'Alembert's system ト云フ

$$y_1 = A_1 e^{\rho x}, \quad y_2 = A_2 e^{\rho x}, \quad y_3 = A_3 e^{\rho x} \text{ トス}$$

$$\left.\begin{aligned} \rho A_1 &= a_1 A_1 + a_2 A_2 + a_3 A_3 \\ \rho A_2 &= b_1 A_1 + b_2 A_2 + b_3 A_3 \\ \rho A_3 &= c_1 A_1 + c_2 A_2 + c_3 A_3 \end{aligned}\right)$$

$$\left.\begin{aligned} (a_1 - \rho)A_1 + a_2 A_2 + a_3 A_3 &= 0 \\ b_1 A_1 + (b_2 - \rho)A_2 + b_3 A_3 &= 0 \\ c_1 A_1 + c_2 A_2 + (c_3 - \rho)A_3 &= 0 \end{aligned}\right)$$

之ヲ解イテ ρ ト A_1, A_2, A_3 ヲ定ム

$$\Delta(\rho) \equiv \begin{vmatrix} a_1 - \rho & a_2 & a_3 \\ b_1 & b_2 - \rho & b_3 \\ c_1 & c_2 & c_3 - \rho \end{vmatrix} = 0 \quad \text{characteristic equation}$$

roots ヲ ρ_1, ρ_2, ρ_3 ト仮定セヨ

[illegible] eq. ヲ [illegible] eq. [illegible] eq.

[illegible] A_1, A_2, A_3 ヲ定メル

$$\begin{array}{c|c|c|c|c} \rho_1 & A_1^{(1)} e^{\rho_1 x}, & A_2^{(1)} e^{\rho_1 x}, & A_3^{(1)} e^{\rho_1 x} & C_1 \\ \hline \rho_2 & A_1^{(2)} e^{\rho_2 x}, & A_2^{(2)} e^{\rho_2 x}, & A_3^{(2)} e^{\rho_2 x} & C_2 \\ \hline \rho_3 & A_1^{(3)} e^{\rho_3 x}, & A_2^{(3)} e^{\rho_3 x}, & A_3^{(3)} e^{\rho_3 x} & C_3 \\ & y_1 \downarrow + & y_2 \downarrow + & y_3 \downarrow + & \end{array}$$

general sol.

rigid body [illegible] small vibration, vortex motion [illegible]
[illegible] x, y, z [illegible] t [illegible] linear f [illegible] velocity [illegible]
ρ_1, ρ_2, ρ_3 [illegible] real [illegible] imaginary [illegible]
[illegible] real [illegible] periodic sol. [illegible]
[illegible] conjugate imaginary [illegible]
$e^{\rho t}$, $\rho > 0$ [illegible] unstable [illegible]
stable [illegible] planetary system [illegible]
$\rho \leqq 0$ [illegible] complex roots [illegible] $\alpha + \beta i$ [illegible]
α real part $\leqq 0$ [illegible]

Integration by infinite Series.
[illegible] second order [illegible] homogeneous eq. [illegible]

$$y'' + p(x)\,y' + q(x)\,y = 0$$

[illegible] particular sol [illegible]
[illegible] sol. [illegible] x [illegible] power series [illegible] expansible [illegible]
assume [illegible]

$$y = c_0 x^{\rho} + c_1 x^{\rho+1} + c_2 x^{\rho+2} + \cdots + c_n x^{\rho+n} + \cdots$$

[illegible] $$y = c_0 x^{\rho} + c_1 x^{\rho-1} + c_2 x^{\rho-2} + \cdots + c_n x^{\rho-n} + \cdots$$

[illegible] y', y'' [illegible] eq. [illegible] eq. [illegible]
[illegible] x [illegible] power [illegible] C [illegible]
[illegible] infinite series [illegible] convergent [illegible]
[illegible]

[illegible] convergence [illegible] x [illegible] depend [illegible] x [illegible]
[illegible] convergence [illegible] domain [illegible] x [illegible]
[illegible] sol. [illegible]

Gauss's diff. eq.

$$y'' + \frac{\gamma - (\alpha+\beta+1)x}{x(1-x)}\,y' - \frac{\alpha\beta}{x(1-x)}\,y = 0 \qquad \alpha, \beta, \gamma \text{ Const.}$$

$$\begin{array}{l|l}
-\alpha\beta & y = c_0 x^{\rho} + c_1 x^{\rho+1} + c_2 x^{\rho+2} + \cdots + c_n x^{\rho+n} + \cdots \\
\gamma-(\alpha+\beta+1)x & y' = c_0\rho x^{\rho-1} + c_1(\rho+1)x^{\rho} + c_2(\rho+2)x^{\rho+1} + \cdots + c_n(\rho+n)x^{\rho+n-1} + \cdots \\
x - x^2 & y'' = c_0\rho(\rho-1)x^{\rho-2} + c_1(\rho+1)\rho x^{\rho-1} + c_2(\rho+2)(\rho+1)x^{\rho} + \cdots + c_n(\rho+n)(\rho+n-1)x^{\rho+n-2} + \cdots
\end{array}$$

$x^{\rho-1}$: $c_0\gamma\rho + c_0\rho(\rho-1)$

x^{ρ} : $-c_0\alpha\beta - c_0(\alpha+\beta+1)\rho - c_0\rho(\rho-1) + c_1\gamma(\rho+1)$

$- -$

$x^{\rho+n}$: $-c_n\alpha\beta - (\alpha+\beta+1)c_n(\rho+n) + \gamma c_{n+1}(\rho+n+1) - c_n(\rho+n)(\rho+n-1)$
$+ c_{n+1}(\rho+n+1)(\rho+n)$

$c_0\rho(\rho+\gamma-1) = 0 \qquad c_0 \neq 0 \quad \rho = 0 \text{ or } \rho = 1-\gamma$

$-c_0(\rho+\alpha)(\rho+\beta) + c_1\gamma(\rho+1) = 0 \qquad c_1 = \dfrac{(\rho+\alpha)(\rho+\beta)}{(\rho+\gamma)(\rho+1)}c_0$

$- - -$

$-c_n(\rho+n+\alpha)(\rho+n+\beta) + c_{n+1}(\rho+n+1)(\rho+n+\gamma) = 0 \qquad c_{n+1} = \dfrac{(\rho+n+\alpha)(\rho+n+\beta)}{(\rho+n+1)(\rho+n+\gamma)}c_n$

1) $\rho = 0 \qquad c_1 = \dfrac{\alpha\beta}{1.\gamma}c_0 \qquad c_2 = \dfrac{(\alpha+1)(\beta+1)}{2.(\gamma+1)}c_1 = \dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}c_0$

$$c_{n+1} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}c_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n)\,\beta(\beta+1)\cdots(\beta+n)}{1\cdot2\cdot3\cdots(n+1)\,\gamma(\gamma+1)\cdots(\gamma+n)}$$

C_0 … homogeneous eq. [illegible] / 1. [illegible]

part. sol.

$y_1 = F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta\cdots(\beta+n-1)}{1\cdot2\cdots n\,\gamma\cdot(\gamma+1)\cdots(\gamma+n-1)}x^n$

Hypergeometric series

2) $\rho = 1-\gamma$ [illegible] α → $\alpha+1-\gamma$, β → $\beta+1-\gamma$,
γ → $2-\gamma$ [illegible]

$y_2 = F(\alpha-\beta+1, \beta-\gamma+1, 2-\gamma; x)\cdot x^{1-\gamma}$

general sol.

$y = C_1y_1 + C_2y_2 = C_1F(\alpha\beta\gamma;x) + C_2x^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x)$

hypergeom. series [illegible] series [illegible] special cases [illegible]

$(1+x)^m = F(-m, \beta, \beta; -x)$

$\log(1+x) = xF(1, 1, 2; -x)$

$e^x = F(1, \beta, 1; \frac{x}{\beta})$ for $\beta \to \infty$

$\cosh x = F(\alpha, \beta, \frac{1}{2}; \frac{x^2}{4\alpha\beta})$ for $\alpha, \beta \to \infty$

此 series: $|x|<1$ convergent.
$|x|>1$ divergent

$|x|>1$ [illegible] 此 series [illegible]
[illegible] $x=0$ [illegible] $x = \infty$ [illegible]

$\frac{1}{x}$ [illegible] expand in descending power [illegible]

$y = C_0x^\rho + C_1x^{\rho-1} + C_2x^{\rho-2} + \cdots + C_nx^{\rho-n} + \cdots$

$y' = C_0\rho x^{\rho-1} + C_1(\rho-1)x^{\rho-2} + \cdots + C_n(\rho-n)x^{\rho-n-1} + \cdots$

$y'' = C_0\rho(\rho-1)x^{\rho-2} + C_1(\rho-1)(\rho-2)x^{\rho-3} + \cdots + C_n(\rho-n)(\rho-n-1)x^{\rho-n-2} + \cdots$

$x^{\rho-n}$: $-\alpha\beta C_n + C_{n-1}(\rho-n+1)\gamma - C_n(\rho-n)(\alpha+\beta+1) - C_n(\rho-n)(\rho-n-1)$
$+ C_n(\rho-n)(\rho-n-1)$
$= C_{n-1}(\rho-n+1)(\rho-n+\gamma) - C_n(\rho-n+\alpha)(\rho-n+\beta)$

x^ρ : $C_0(\rho+\alpha)(\rho+\beta)$

$\rho = -\alpha$ or $-\beta$.

$\frac{C_n}{C_{n-1}} = \frac{(\rho-n+1)(\rho-n+\gamma)}{(\rho-n+\alpha)(\rho-n+\beta)}$

1) $\rho = -\alpha$ $\frac{C_n}{C_{n-1}} = \frac{(\alpha+n-1)(\alpha+n-\gamma)}{n(\alpha+n-\beta)} = \frac{(\alpha+n-1)(\alpha-\gamma+1+n-1)}{n(\alpha-\beta+1+n-1)}$

$y_1 = x^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; \frac{1}{x})$

2) $\rho = -\beta$.
$y_2 = x^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1; \frac{1}{x})$

$y = C_1y_1 + C_2y_2$

[illegible] $|x|$ [illegible] 此 [illegible] convergency [illegible]
[illegible] $|x|$ [illegible]

$|x|$ [illegible] convergency [illegible]
[illegible] $|x| = 1-\varepsilon$ or $|x| = 1+\varepsilon$ [illegible]
[illegible]

$$x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0$$

$$x = 1-t$$

$$y' = \frac{dy}{dx} = -\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$$

$$t(1-t)\frac{d^2y}{dt^2} + \left\{(\cancel{\gamma -} \alpha+\beta+1 - \gamma) - (\alpha+\beta+1)t\right\}\frac{dy}{dt} - \alpha\beta y = 0$$

[illegible] γ [illegible] $\alpha+\beta+1-\gamma$ [illegible]

$$y_1 = F(\alpha, \beta, \alpha+\beta+1-\gamma; \underset{\overset{\|}{1-x}}{t})$$

$$y_2 = t^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta; \underset{\overset{\|}{1-x}}{t})$$

t [illegible] expand [illegible] $1-x = t$ [illegible]
[illegible]

[illegible] particular sol [illegible]
$1-\frac{1}{x}$, $\frac{1}{1-x}$, $\frac{1}{1-\frac{1}{x}}$, $1-\frac{1}{1-x}$ [illegible] expand [illegible]
[illegible]

Legendre's differential equati[on]

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1)y = 0$$

or $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\begin{array}{r|l} n(n+1) & y = C_0 x^{\rho} + C_1 x^{\rho-1} + \cdots + C_n x^{\rho-n} + \cdots \\ -2x & y' = C_0\rho x^{\rho-1} + C_1(\rho-1)x^{\rho-2} + \cdots + C_n(\rho-n)x^{\rho-n-1} + \cdots \\ (1-x^2) & y'' = C_0\rho(\rho-1)x^{\rho-2} + C_1(\rho-1)(\rho-2)x^{\rho-3} + \cdots + C_n(\rho-n)(\rho-n-1)x^{\rho-n-2} + \cdots \end{array}$$

x^{ρ} : $n(n+1)C_0 - 2C_0\rho - C_0\rho(\rho-1) = C_0\{n(n+1) - \rho(\rho+1)\}$

$x^{\rho-1}$: $n(n+1)C_1 - 2(\rho-1)C_1 - (\rho-1)(\rho-2)C_1 = C_1\{n(n+1) - (\rho-1)\rho\}$

$\cdots$

$x^{\rho-m}$: $n(n+1)C_m - 2(\rho-m)C_m - (\rho-m)(\rho-m-1)C_m + C_{m-2}(\rho-m+2)(\rho-m+1)$

$= C_m\{n(n+1) - (\rho-m)(\rho-m+1)\} + C_{m-2}\{\rho-m+2)(\rho-m+1)$

$C_0\{n(n+1) - \rho(\rho+1)\} = C_0(n-\rho)(n+\rho+1) = 0$

$\rho = n$ or $\rho = -(n+1)$ [illegible] $C_0 \neq 0$

then $n(n+1) - (\rho-1)\rho \neq 0$

$\therefore\ C_1 = 0$

$$\frac{C_m}{C_{m-2}} = -\frac{(\rho-m+1)(\rho-m+2)}{n(n+1) - (\rho-m)(\rho-m+1)}$$

1). $\rho = n$

$$\frac{C_m}{C_{m-2}} = -\frac{(n-m+1)(n-m+2)}{m(2n-m+1)}, \quad C_1 = 0$$

$\therefore\ C_1 = 0\ \ C_3 = 0\ \cdots\ C_{2k+1} = 0$

$$y = C_0\left\{x^n - \frac{n(n+1)}{2(2k-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

n [illegible] positive integer [illegible]
[illegible]

$$\frac{C_m}{C_{m-2}} = \frac{\{n-(m-1)\}\{n-(m-2)\}}{\ldots}$$ となるから 分子は 0 となり finite series となる

n even 2ν とし

$2\nu = m-2 \qquad m = 2\nu+2 = n+2$

$C_{n+2} = C_{n+4} = \cdots = 0$

x^0 の term になるのは $C_{n+2}\, x^{\rho-n-2}$ だから 0

n odd なら

x^1 が last term とし

故に y_1 は x の n th degree の polynomial となる

此の場合 $C_0 = \dfrac{(2n)!}{2^n n! n!}$ と置くことにする

これを nth Legendre's function or polynomial という

$P_n(x)$ 又 $X_n(x)$ とかき 又これを nth zonal harmonics ともいう

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$ に等しい。

故に n を正の整数とする eq. は Legendre's polynomial なる一の particular sol. とし

2). $\rho = -(n+1)$

$$\frac{C_{m+2}}{C_m} = \frac{(n+m+1)(n+m+2)}{(m+2)(2n+m+3)}$$

$c_1 = 0$

$$y_2 = C_0\left\{ x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)}x^{-(n+5)} + \cdots \right\}$$

Convergency を考える

1. の場合 $$\left|\frac{1}{x^2}\frac{C_{m+2}}{C_m}\right| = -\frac{(n-m)(n-m-1)}{(m+2)(2n-m-1)}\frac{1}{x^2}$$ test ratio

2. $$\frac{C_{m+2}}{C_m} = \frac{(n+m+1)(n+m+2)}{(m+2)(2n+m+3)}$$

$m \to \infty$ の limit をとる

1. の場合 $$\frac{1}{|x^2|}\left|\frac{(m-n)(m+1-n)}{(m+2)(m+1-2n)}\right|$$

$$= \frac{1}{|x^2|}\left|\frac{\left(1-\frac{n}{m}\right)\left(1+\frac{1-n}{m}\right)}{\left(1+\frac{2}{m}\right)\left(1+\frac{1-2n}{m}\right)}\right|$$

$$\lim_{m\to\infty}|\;| = \frac{1}{|x^2|} < 1 \quad \text{conv.}$$

$$> 1 \quad \text{div.}$$

2. の場合も同様に

故に一般に $|x| > 1$ で conv. となる

n が ~~polyno~~ positive integer ならば

y_2 は $C_0 = \dfrac{2^n n! n!}{(2n+1)!}$ とおいて $Q_n(x)$ とする

此は polynomial にならない

n が negative integer ならば

y_2 の [illegible] は 0 となり finite series となる

y_1 は infinite series となる

[illegible] n の [illegible] に [illegible]

1° $2n$ = odd positive integer $= 2\nu+1$

$y_1: \quad \frac{C_{m+2}}{C_m} = -\frac{(n-m)(n-m-1)}{(m+2)(2n-m-1)}$

$(m+2)(2n-m-1)C_{m+2} + (n-m)(n-m-1)C_m = 0$

$2n$ が $m+1$ [illegible] (m even)

$(0)\, C_{2\nu+2} + (\ )(\ )\, C_{2\nu} = 0$

$C_{2\nu} = 0$ [illegible]

$C_{2\nu+2}$ は 不定

故に $C_{2\nu} = 0 \quad C_{2\nu-2} = 0 \quad \cdots \quad C_2 = 0 \quad C_0 = 0$

$C_{2\nu+2}$ [illegible] arbitrary [illegible] $C_{2\nu+4}$ 以下 [illegible]

$y_1 = C_{2\nu+2}\left\{x^{n-(2\nu+2)} - \frac{(n-2\nu-2)(n-2\nu-3)}{(2\nu+4)(2n-2\nu-5)}x^{n-(2\nu+4)} + \cdots\right\}$

$2\nu+1 = 2n$

$y_1 = C_{2n+1}\left\{x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \cdots\right\}$

此 y_1 [illegible]

$m = 2\nu+2+2k$

$\frac{C_{m+2}}{C_m} = \frac{(n+2k+1)(n+2k+2)}{(2k+2)(2n+2k+3)}$

[illegible] C_m (k+1)th term [illegible] C_{m+2} [illegible]

[illegible] y_2 の $\frac{C_{m+2}}{C_m} = \frac{(n+m+1)(n+m+2)}{(m+2)(2n+m+3)}$

此 $m = k+1$ [illegible]

[illegible] y_1 と y_2 [illegible] (constant [illegible])

linearly dependent [illegible] fundamental system [illegible]

$y_1 = \text{Const.} \times y_2$

2° $2n$ = odd negative integer $(\neq 1) = -(2\nu+1)$, $\nu \neq 0$

y_1 [illegible] 0 [illegible] y_2 [illegible] 0 [illegible]

[illegible] $y_2 = \text{const.} \times y_1$

3° $2n = -1$, $n = -(n+1)$ $\quad p = n = -(n+1)$

$y_1 = y_2$

此 3 cases [illegible] y_1 と y_2 は linearly independent [illegible] case [illegible]

~~[illegible]~~ 2 [illegible] $p = n = -(n+1)$ [illegible] y_1, y_2 [illegible]

0 0 [illegible]

[illegible] case [illegible] particular sol. と

linearly indep. の sol. [illegible]

y_1 part. sol. [illegible] $y_1\int \frac{e^{-\int p\,dx}}{y_1^2}dx$ [illegible]

(homog. linear eq. of second order)

p [illegible] $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

$p(x) = -\frac{2x}{1-x^2}$

$e^{-\int p\,dx} = \frac{1}{1-x^2}$

故ニ $\quad y_1 \quad , \quad y_1 \int \frac{dx}{y_1^2(1-x^2)}$

然シテ y_1 ハ infinite series ナル故ニ impracticable ナリ 他ノ方法ニ依リテ [illegible]

put q, $y = v + u y_1$

$p \quad y' = v' + u' y_1 + u y_1'$

$y'' = v'' + u'' y_1 + 2 u' y_1' + u y_1''$

$0 = v'' + p v' + q v + y_1 (u'' + p u') + 2 u' y_1'$

$u'' + p(x) u' = 0$ [illegible] u ヲ [illegible]

$v'' + p v' + q v = -2 u' y_1'$

$u' = e^{-\int p(x) dx} \qquad \therefore \frac{u''}{u'} = -p(x) \quad \log u' = -\int p\, dx$

$= \frac{1}{1-x^2} \qquad p(x) = \frac{-x}{1-x^2}$

$u = \frac{1}{2} \log \frac{x+1}{x-1}$

$\cancel{(1-x^2)}\, v'' + \overset{2x}{\cancel{p}} v' + \overset{n(n+1)}{\cancel{q}} v = -\frac{2}{1-x^2} y_1'$

$(1-x^2) v'' - 2 x v' + n(n+1) v = -2 y_1'$

$v = c_0 x^{\rho} + c_1 x^{\rho-1} + c_2 x^{\rho-2} + \cdots$

$y_1 = x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} + \cdots$

$y_1' = n \left(x^{n-1} - \frac{(n-1)(n-2)}{2(2n-1)} x^{n-3} + \cdots \right)$

[illegible] v ノ [illegible] ρ ハ y_1' ノ highest degree ト一致セシメ

Coefficient $c_0, c_1 \cdots$ ヲ定メ

$y_2 = v + \frac{1}{2} \log \left(\frac{x+1}{x-1} \right) y_1$

Bessel's differential equation

$x^2 y'' + x y' + (x^2 - \lambda^2) y = 0$

$x^2 - \lambda^2 \quad y = c_0 x^{\rho} + c_1 x^{\rho+1} + \cdots + c_n x^{\rho+n} + \cdots$

$x \quad y' = c_0 \rho x^{\rho-1} + c_1 (\rho+1) x^{\rho} + \cdots + c_n (\rho+n) x^{\rho+n-1} + \cdots$

$x^2 \quad y'' = c_0 \rho(\rho-1) x^{\rho-2} + c_1 (\rho+1) \rho x^{\rho-1} + \cdots c_n (\rho+n)(\rho+n-1) x^{\rho+n-2} + \cdots$

$x^{\rho}: \quad -c_0 \lambda^2 + c_0 \rho + c_0 \rho(\rho-1) = c_0 (\rho^2 - \lambda^2) = 0$

$x^{\rho+1}: \quad -c_1 \lambda^2 + c_1 (\rho+1) + c_1 (\rho+1) \rho = c_1 ((\rho+1)^2 - \lambda^2) = 0$

$x^{\rho+n}: \quad -c_n \lambda^2 + c_{n-2} + c_n (\rho+n) + c_n (\rho+n)(\rho+n-1)$

$= c_n ((\rho+n)^2 - \lambda^2) + c_{n-2} = 0$

$\rho^2 = \lambda^2$

$c_1 = 0$

$c_n = -\frac{c_{n-2}}{(\rho+n)^2 - \lambda^2} = -\frac{c_{n-2}}{n(2\rho+n)}$

1) $\rho = +\lambda$.

$c_1 = 0$

$c_n = -\frac{c_{n-2}}{n(2\lambda+n)}$

$y_1 = c_0 x^{\lambda} \left\{ 1 - \frac{x^2}{2^2(\lambda+1)} + \frac{x^4}{2! 2^4 (\lambda+1)(\lambda+2)} - \frac{x^6}{3! 2^6 (\lambda+1)(\lambda+2)(\lambda+3)} + \cdots \right\}$

$= c_0 x^{\lambda} \left\{ 1 - \frac{x^2}{2(2\lambda+2)} + \frac{x^4}{2 \cdot 4 (2\lambda+2)(2\lambda+4)} - \frac{x^6}{2 \cdot 4 \cdot 6 (2\lambda+2)(2\lambda+4)(2\lambda+6)} + \cdots \right\}$

2) $\rho = -2$

$c_1 = 0$

$c_n = -\frac{c_{n-2}}{n(-2\lambda+n)}$

$y_2 = c_0 x^{-\lambda} \left\{ 1 - \frac{x^2}{2(-2\lambda+2)} + \frac{x^4}{2 \cdot 4 (-2\lambda+2)(-2\lambda+4)} - \cdots \right\}$

$c_0 = \frac{1}{2^{\lambda} \Gamma(\lambda+1)}$

$C_0 = \frac{1}{2^{\lambda}\Gamma(\lambda+1)}$

此時（？）

$J_{-\lambda}(x)$ トナ（？）

$\Gamma(\lambda+1)$ … λ が integer ナラ…

$\lambda!$ … Gammafunct.

$J_\lambda(x)$ Bessel's funct.

(Zylinderfunktion)

General sol. …

$y = C_1 J_\lambda(x) + C_2 J_{-\lambda}(x)$

$$\left|\frac{C_n}{C_{n-2}}x^2\right| = |x^2|\frac{1}{|n(2p+n)|} \to 0 \text{ when } n \to \infty$$

x = 如何（？）… convergent.

y_1 ~~相~~ … 0 トナル … λ が negative integer

y_2 〃 positive integer ノトキ…

$\lambda = -n$; (n posit. integer)

$$C_m = -\frac{C_{m-2}}{m(2\lambda+m)}$$

$$C_{m-2} - C_m\, m(m-2n) = 0.$$

$m = 2n$ $\quad C_{2n-2} = 0$

$C_{2n-4} = 0$

⋮

$C_2 = 0$

$C_0 = 0$

$$y_1 = C_{2n}x^n\left\{1 - \frac{x^2}{2(n+2)} + \frac{x^4}{2\cdot4(n+2)(n+4)} - \frac{x^6}{2\cdot4\cdot6(n+2)(n+4)(n+6)}\right\}$$

此時（？） y_2 卜 const. …

$y_1 = \text{Const.} \times y_2$

…故ニ $\lambda = n$ ノトキ …

$\lambda = 0 \quad y_1 = y_2$

故ニ … y …

$y = v + u\, J_n(x)$

$u' = e^{-\int p\,dx} \qquad p = \frac{x}{x^2} = \frac{1}{x}$

$= \frac{1}{x}$

$u = \log x$

$y = \log x \cdot J_n(x) + v$

$x^2 v'' + x v' + (x^2 - n^2)v = -2x J_n'(x)$

$Y_n(x) = \log x \cdot J_n(x) + v$

ヲ Bessel's f. of the second kind トス

v … Bessel's f. …

Convergent series ト convergency ガ … divergent …

Asymptotic series

x ガ大ナル…

$a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots + \frac{a_n}{x^n} + \dots$

$\frac{1}{x}$ … divergent …

$f(x) \qquad \lim_{x\to\infty}\{f(x) - a_0\} = 0$

$$\lim_{x\to+\infty} x\left\{f(x) - a_0 - \frac{a_1}{x}\right\} = 0$$

$$\lim_{x\to+\infty} x^2\left\{f(x) - a_0 - \frac{a_1}{x} - \frac{a_2}{x^2}\right\} = 0$$

$$\lim_{x\to+\infty} x^n\left\{f(x) - a_0 - \frac{a_1}{x} - \frac{a_2}{x^2} - \cdots - \frac{a_n}{x^n}\right\} = 0$$

ナルトキ $f(x)$ ハ此ノ divergent series ニ依リ asymptotically ニ表ハサルト云フ

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + \cdots$$

此ノ div. s. ヲ $f(x)$ ノ asymptotic series ト云フ 又ハ semi-convergent series トイフコトアリ semi-conv. ハ conditionally conv. ノ意ニ用ヰル故 semi-conv. ノ名ハ良クナイ

上ノ式ハ

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + \frac{r_n(x)}{x^n}$$

トスレバ $\lim_{x\to+\infty} r_n(x) = 0$ ナルヲ意味ス

故ニ x 大ナルトキ $f(x)$ ハ $a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}$ ニテ approximately ニ表ハサレル infinite series トシテハ divergent ナレド n ヲ一定ニシテ x ヲ大キクスレバ $f(x)$ ノ値ハ $a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}$ ニテ表ハサル

Ex. integral logarithm

$$J = -li(e^{-x}) = +\int_x^\infty \frac{e^{-u}}{u}\,du \qquad li \text{ ハ } \int \frac{dx}{\log x},\ x = e^{-x} \text{ トス}$$

之ハ conv. s. ニヨリテハ asympt. s. ニ因ル

$$u = x(1+t)$$

$$J = e^{-x}\int_0^\infty \frac{e^{-xt}}{1+t}\,dt$$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1}t^{n-1} + (-1)^n\frac{t^n}{1+t}$$

$$J = e^{-x}\left\{\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n} + \frac{r_n(x)}{x^n}\right\}$$

$$r_n(x) = (-1)^n\int_0^\infty \frac{t^n e^{-xt}}{1+t}\,dt$$

$$\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n} + \cdots$$

divergent series

$$\because \text{ test ratio } \quad \frac{\frac{(-1)^n n!}{x^{n+1}}}{\frac{(-1)^{n-1}(n-1)!}{x^n}} = -\frac{n}{x} \to \infty \text{ when } x\to\infty$$

$$x^n\left\{e^x J - \left(\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n}\right)\right\} = r_n(x)$$

$$|r_n(x)| = x^n\int_0^\infty \frac{t^n e^{-xt}}{1+t}\,dt < x^n\int_0^\infty t^n e^{-xt}\,dt = x^n\frac{n!}{x^{n+1}} = \frac{n!}{x}$$

n ヲ一定ニシテ x ヲ大キクスレバ

$$\lim_{x\to+\infty} |r_n(x)| = 0$$

$$\therefore\ e^x J \sim \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots$$

$$J \sim e^{-x}\left(\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots\right)$$

J [illegible] series [illegible] 値 [illegible] 計算 [illegible]
[illegible]

$x=10$, $n=9$ [illegible] 値 [illegible] conv. [illegible]
[illegible]

[illegible] dif. eq. [illegible] J [illegible] y [illegible]

$$y = \int_x^\infty \frac{e^{-u}}{u}\,du$$

$$y' = -\frac{e^{-x}}{x}$$

$$xy' = -e^{-x}$$

$$xy'' + y' = e^{-x} = -xy'$$

y [illegible] dif. eq. [illegible] y [illegible] coeff [illegible] polynomial
[illegible]

$$xy'' + (1+x)y' = 0$$

$$y'' + \left(1 + \frac{1}{x}\right)y' = 0$$

$x \to +\infty$ $\quad y'' + y' = 0$

$$y' = e^{-x}$$

[illegible] $y'' + \left(1+\frac{1}{x}\right)y' = 0$ [illegible] sol. [illegible] $y' = e^{-x}$ [illegible] $y = e^{-x}$

$y = e^{-x}z$ [illegible]

$$y' = e^{-x}z' - e^{-x}z = -\frac{e^{-x}}{x}$$

$$z' - z = -\frac{1}{x}$$

$$1 + xz' - xz = 0$$

$$xz'' + z' - xz' - z = 0$$

$$xz'' + (1-x)z' - z = 0$$

$$z = c_0x^{\rho} + c_1x^{\rho-1} + c_2x^{\rho-2} + \cdots + c_nx^{\rho-n} + \cdots$$

$$z' = c_0\rho x^{\rho-1} + c_1(\rho-1)x^{\rho-2} + c_2(\rho-2)x^{\rho-3} + \cdots + c_n(\rho-n)x^{\rho-n-1} + \cdots$$

$z' - z = -\frac{1}{x}$ [illegible]

$$-c_0x^{\rho} + (c_0\rho - c_1)x^{\rho-1} + \cdots + (c_{n-1}(\rho-n+1) - c_n)x^{\rho-n} + \cdots = -\frac{1}{x}$$

$c_0 = 1 \quad \rho = -1$

$c_0\rho - c_1 = 0 \qquad c_1 = -1$

$- - - - \qquad c_2 = -2c_1 = +2$

$c_{n-1}(\rho-n+1) - c_n = 0 \qquad c_n = \quad c_{n-1} = (-1)^n n!$

$$z_1 = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n} + \cdots$$

general sol. [illegible] $z' - z = 0$ [illegible] $ce^x + z_1$ [illegible]

$$z = ce^x + z_1$$

$$y = c + e^{-x}z_1$$

$x \to \infty$ [illegible] $y = 0$ [illegible] $c = 0$

$$y = e^{-x}z_1$$

[illegible] $y \sim e^{-x}z_1$ div. [illegible] asymptotic

Asymptotic series [illegible] 19th [illegible] Poincaré [illegible]
[illegible] dif. eq. [illegible] asympt. [illegible]
[illegible] homog. eq. [illegible] coeff [illegible] x
polynomial [illegible] dif. eq. [illegible] asympt. [illegible]
[illegible] assume [illegible] ([illegible])

Gauss's Error funct. [illegible]

$$f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{x}{1} - \frac{x^3}{1!\,3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \cdots\right)$$

x [illegible] converge [illegible]

[illegible] asymptotic series [illegible]

$$y = \int_0^x e^{-x^2}\,dx$$

$$y' = e^{-x^2}$$

put $y = e^{-x^2} z \qquad y' = e^{-x^2} z' - 2x e^{-x^2} z = e^{-x^2}$

$$\therefore \quad z' - 2xz - 1 = 0$$

$$z'' - 2xz' - 2z = 0$$

linear homog [illegible] coef. [illegible] x [illegible] polynomial [illegible] asymptotic series [illegible]

$$\begin{array}{r|l} -2x & z = c_0 x^{\rho} + c_1 x^{\rho-1} + c_2 x^{\rho-2} + \cdots + c_n x^{\rho-n} + \cdots \\ 1 & z' = c_0\rho x^{\rho-1} + c_1(\rho-1)x^{\rho-2} + c_2(\rho-2)x^{\rho-3} + \cdots + c_n(\rho-n)x^{\rho-n-1} + \cdots \end{array}$$

$$1 = -2c_0 x^{\rho+1} - 2c_1 x^{\rho} + (c_0\rho - 2c_2)x^{\rho-1} + \cdots + (c_n(\rho-n) - 2c_{n+2})x^{\rho-n-1} + \cdots$$

$$\rho = -1, \quad c_0 = -\frac{1}{2}, \quad c_1 = 0, \quad c_2 = \frac{c_0\rho}{2} = \frac{1}{2\cdot 2}, \ \cdots$$

$$c_{n+2} = \frac{c_n(\rho-n)}{2} = -\frac{c_n(n+1)}{2}$$

$$z_1 = -\frac{1}{2x}\left\{1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \cdots\right\}$$

$z' - 2xz = 0 \qquad \frac{z'}{z} = 2x \qquad \log z = x^2 + c \qquad z = Ce^{x^2}$

$$\therefore \quad z = z_1 + Ce^{x^2}$$

$$y = e^{-x^2} z = C + e^{-x^2} z_1$$

$x \to \infty \qquad y = \frac{\sqrt{\pi}}{2}, \qquad C = \frac{\sqrt{\pi}}{2}$

$$f(x) = 1 - \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x}\left\{1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \cdots\right\}$$

divergent [illegible] x [illegible]

Ex. 3. Bessel's funct.

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

$$\begin{array}{r|l} x^2 - n^2 & y = uv \\ x & y' = uv' + u'v \\ x^2 & y'' = uv'' + 2u'v' + u''v \end{array}$$

$$x^2 u v'' + (2x^2 u' + xu)v' + (\ )v = 0$$

v' [illegible] coeff 0 [illegible] u [illegible]

$$\frac{u'}{u} = -\frac{1}{2x} \qquad \log u = -\frac{1}{2}\log x + c, \qquad u = \frac{1}{\sqrt{x}}$$

$$v'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right)v = 0$$

$x \to \infty$ [illegible] $v'' + v = 0 \qquad v = C_1\cos x + C_2\sin x$ [illegible] sol.

x [illegible] sol. [illegible]

put $v = \cos x\left\{a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right\} + \sin x\left\{b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\right\}$

$$\begin{aligned} v' = {} & \cos x\left\{-\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \cdots - \frac{n a_n}{x^{n+1}} - \cdots\right\} \\ & + \sin x\left\{-\frac{b_1}{x^2} - \frac{2b_2}{x^3} - \cdots - \frac{n b_n}{x^{n+1}} - \cdots\right\} \\ & - \sin x\left\{a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right\} \\ & + \cos x\left\{b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\right\} \\ = {} & \cos x\left\{b_0 + \frac{b_1}{x} + \frac{(b_2 - a_1)}{x^2} + \cdots + \frac{(b_n - (n-1)a_{n-1})}{x^n} + \cdots\right\} \\ & - \sin x\left\{a_0 + \frac{a_1}{x} + \frac{(a_2 + b_1)}{x^2} + \cdots + \frac{a_n + (n-1)b_{n-1}}{x^n} + \cdots\right\} \end{aligned}$$

$$v'' = -\cos x\left\{a_0 + \frac{a_1}{x} + \frac{(a_2+2b_1)}{x^2} + \sum_{n=3}^{\infty}\left(a_n + 2(n-1)b_{n-1} - (n-2)(n-1)a_{n-2}\right)\frac{1}{x^n}\right\}$$
$$- \sin x\left\{b_0 + \frac{b_1}{x} - \frac{b_2+2a_1}{x^2} - \sum_{n=3}^{\infty}\left(-b_n + 2(n-1)a_{n-1} + (n-1)(n-2)b_{n-2}\right)\frac{1}{x^n}\right\}$$

$\left(1+\frac{1-4m^2}{4x^2}\right)v + v'' = 0$ ヨリ

$\cos x$, $\sin x$ ノ各々ノ coeff. ヲ比較シ x ノ power ニ equate

$a_2 + a_0\frac{(1-4m^2)}{4} = a_2 + 2b_1$, $b_2 + b_0\frac{1-4m^2}{4} = b_2 - 2a_1$

$a_n + a_{n-2}\frac{1-4m^2}{4} = a_n + 2(n-1)b_{n-1} - (n-1)(n-2)a_{n-2}$

$b_n + b_{n-2}\frac{1-4m^2}{4} = b_n - 2(n-1)a_{n-1} - (n-1)(n-2)b_{n-2}$

$$\begin{cases}(\quad)\,a_{n-2} = b_{n-1}(\quad)\\ (\quad)\,b_{n-2} = a_{n-1}(\quad)\end{cases}$$ ヨリ [illegible] a ノ b ノ [illegible]

a_0, b_0 arbitrary constants

$v = \cos x\left(a_0\varphi_1(x) + b_0\varphi_2(x)\right) + \sin x\left(b_0\varphi_1(x) - a_0\varphi_2(x)\right)$

$\varphi_1(x)$, $\varphi_2(x)$ ハ series

$$\varphi_1(x) = 1 - \frac{\left(\left(\frac{1}{2}\right)^2-n^2\right)\left(\left(\frac{3}{2}\right)^2-n^2\right)}{2^2\cdot 2!}\frac{1}{x^2} + \frac{\left(\left(\frac{1}{2}\right)^2-n^2\right)\left(\left(\frac{3}{2}\right)^2-n^2\right)\left(\left(\frac{5}{2}\right)^2-n^2\right)\left(\left(\frac{7}{2}\right)^2-n^2\right)}{2^4\cdot 4!}\frac{1}{x^4}$$

$$\varphi_2(x) = \frac{\left(\frac{1}{2}\right)^2-n^2}{2\cdot 1!}\frac{1}{x} + \frac{\left(\left(\frac{1}{2}\right)^2-n^2\right)\left(\left(\frac{3}{2}\right)^2-n^2\right)\left(\left(\frac{5}{2}\right)^2-n^2\right)}{2^3\cdot 3!}\frac{1}{x^3} \cdots$$

a_0, b_0 ハ Bessel f. ノ [illegible]

$J_m(x)$ ニ於テ $a_0 = \sqrt{\frac{2}{\pi}}\cos\frac{2m+1}{4}\pi$

$b_0 = \sqrt{\frac{2}{\pi}}\sin\frac{2m+1}{4}\pi$

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}}\left\{\cos\left(\frac{2m+1}{4}\pi - x\right)\varphi_1(x) + \sin\left(\frac{2m+1}{4}\pi - x\right)\varphi_2(x)\right\}$$

x 大ナルトキ φ_1 φ_2 ノ [illegible]

Partial differential equation of the first order

x, y indep. var. z dep. v. funct. of x, y

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ヲ p, q ト書ク

$F(x, y, z, p, q) = 0$ dif. eq. of the 1st order

space ニテ考フレバ $z = f(x, y)$ ハ space surface

Tangential plane through a, b, c

$Z - c = p(x-a) + q(y-b)$

normal ハ $p : q : -1$ ニ proport. ナル direct. cos.

[illegible] spatial pt. x, y, z ヲ pass スル [illegible] elementary plane ヲ考ヘ其 normal direct. Cosine ヲ $p : q : -1$ トシ

(x, y, z, p, q) ナル elementary plane ヲ surface-element ト云フ

斯ノ如キ surface elements ハ ∞^5 アリ [illegible] x, y, z, p, q ヲ indep. variables トス

故ニ一ノ 1st order ノ part. dif. eq. ガアレバ ∞^5 ノ中ニ [illegible]

∞^4 ノ surface elements ヲ define スル

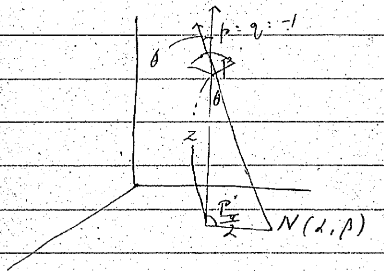
solution ハ $z = f(x, y)$ ヲ [illegible] surface ノ tangential plane ハ ∞^4 ノ中ニ [illegible] surface ヲ [illegible] integral surface ト云フ

一ノ surface ノ上ノ pt. ハ ∞^2 [illegible] 一ノ surface ノ上ノ surface element ハ ∞^2 故ニ ∞^4 ノ中ニ [illegible] surface ノ族 ∞^2 ナリ. 故ニ integral surface ハ二ツノ arbitrary constants ヲ含ム $z = f(x, y, \alpha, \beta)$

例 $z^2(1+p^2+q^2) = a^2$

$$z = \frac{\pm a}{\sqrt{1+p^2+q^2}}$$

$$\frac{1}{\sqrt{1+p^2+q^2}}$$



dir. cosine $\frac{p}{\sqrt{p^2+q^2+1}}, \frac{q}{V}, \frac{-1}{V}$

$$\cos\theta = \pm \frac{1}{\sqrt{1+p^2+q^2}}$$

$$z = a \cos\theta$$

$PN = a$ ナルヲ示ス

normal ガ xy plane ニ交ハル点ノ dist. ガ const. a ナル如キ surface elements ∞^4 ヲ作ル. 其中適当ノモノヲ選ビテ一ノ surface ヲ作レバ sol. ナリ

今任意ニ定メタ PN ナル直線ヲ考ヘルト P ニ於テ normal ノ surface element ノ集ハ N ヲ中心トシ $PN = a$ ヲ radius トスル ~~surface~~ sphere ヲ作ル.

$(x-\alpha)^2 + (y-\beta)^2 + z^2 = a^2$ integral surface,

$\Phi_{(x, y, z, \alpha, \beta)} \equiv (x-\alpha)^2 + (y-\beta)^2 + z^2 - a^2$ ニシテ

Complete solution トナル

次ニ N ヲ xy plane 上ノ任意ノ curve ニ沿ッテ動カス $\beta = \varphi(\alpha)$, φ arbitrary funct. ナル関係アリトセヨ

此 curve ノ上ニ centre ヲ有スル sphere ノ envelope ヲ作レバ其 surface element ハ ∞^4 ノ中ニ含マル (Ring トナル)

$\Phi(x, y, z, \alpha, \varphi(\alpha)) = 0$ ナル ∞^1 ノ sphere ノ envelope ヲ $\frac{\partial\Phi}{\partial\alpha} = 0$ ト $\Phi(x, y, z, \alpha, \varphi(\alpha)) = 0$ トヨリ α ヲ eliminate シテ得ラル. 其結果ハ α ナル arbitrary const. ヲ含マズシテ φ ノ arbitrary function ヲ含ム. 之レ given eq. ヲ satisfy スル surface ナリ. 之ヲ

general solution トス

general sol. ノ外ニ complete sol. ヨリ得ラルル場合アリ. Φ ナル [illegible] sphere ノ envelope ガ [illegible] xy ニ平行シ a ナル dist. ノ plane ナリ

$$z = \pm a$$

之ヲ singular solution トス

$$\begin{cases} \Phi(x, y, z, \alpha, \beta) = 0 \\ \frac{\partial\Phi}{\partial\alpha} = 0 \\ \frac{\partial\Phi}{\partial\beta} = 0 \end{cases}$$

ヨリ α, β ヲ eliminate シ

此ノ外ニ [illegible]

此ヲ一般ニ 1st order part. dif. eq. [illegible] ノ sol. トス

Linear partial diff. eq. of the 1st order.

$P(x\,y\,z)\,p + Q(x\,y\,z)\,q = R(x\,y\,z)$ (1)

p, q ニツキ linear.

[illegible] integral surface [illegible]

$f(x\,y\,z) = 0$ [illegible]

其 tangential plane ノ normal ノ direction cosine

$p : q : -1$ [illegible]

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\,p = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\,q = 0$$

[illegible] p, q [illegible] eq. [illegible]

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = 0 \qquad (2)$$

此 eq. ハ

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$$

$$u = u(x\,y\,z)$$

three variables ノ linear homogeneous eq. [illegible]

[illegible] (1) [illegible] sol. $u = c$ [illegible] (1) ノ

sol. [illegible] $f(x\,y\,z) = c$

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0 \qquad (3)$$

[illegible] (2) [illegible]

dx, dy, dz ハ surface 上ノ pt [illegible] increment [illegible] surface 上ノ line element [illegible]

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (4)$$

$dx : dy : dz = P : Q : R$ ナル line element

此 $\dfrac{dy}{dx} = \dfrac{Q(x\,y\,z)}{P(x\,y\,z)}$, $\dfrac{dz}{dx} = \dfrac{R(x\,y\,z)}{P(x\,y\,z)}$ [illegible]

此 x ナル一ツノ indep. v. ノ simultaneous ordinary dif. eq. [illegible] sol. ハ space curve ナリ

$$\begin{cases} y = \varphi(x, \alpha, \beta) \\ z = \psi(x, \alpha, \beta) \end{cases} \quad (5)$$ [illegible] (4) ノ sol. ナリ

(4) ノ line element ハ $f(x\,y\,z) = c$ ノ surface 上ニアリ

[illegible] space curve ハ此 surface 上ニアリ

此 space curves ノ surface ヲ generate [illegible] eq. ノ sol. [illegible]

(4) ノ eq. ヲ Lagrange's subsidiary equation トス

(5) ヨリ $\alpha = \Phi(x, y, z)$

$\beta = \Psi(x, y, z)$

此 [illegible] space curve [illegible]

surface [illegible] α, β [illegible] co', co' ノ

surface [illegible] (4) ノ eq. [illegible] 此 eq. ハ

(2) ノ sol. [illegible]

$\beta = u(\alpha)$ u ハ any funct.

$\Psi = u(\Phi)$ [illegible] surface ナリ

[illegible] $\chi(\alpha\,\beta) = 0$ $\chi(\Phi(x\,y\,z), \Psi(x\,y\,z)) = 0$

[illegible] general solution ナリ

Ex. $xp + yq = z$

$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ subsid. eq.

$\frac{dy}{dx} = \frac{y}{x}, \quad \frac{dz}{dx} = \frac{z}{x}$

$\log x = c + \log y$

$\frac{y}{x} = \alpha \qquad \frac{z}{x} = \beta$

$\begin{cases} y = \alpha x \\ z = \beta x \end{cases}$ co², space curves

origin pass all straight lines

origin vertex cones surfaces

$y = \alpha x$. z plane

$z = \beta x$ y " "

$\chi(\alpha, \beta) = 0 \quad \chi\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ or $\frac{z}{x} = \chi\left(\frac{y}{x}\right)$

origin vertex cones eq.

vertex $(a\,b\,c)$

$(x-a)p + (y-b)q = z - c$ eq.

---

f general sol. ($P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$)

$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = 0$

$P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y} + R\frac{\partial \phi}{\partial z} = 0$

$P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\frac{\partial \psi}{\partial z} = 0$

$\frac{\partial(f, \phi, \psi)}{\partial(x, y, z)} = 0$ identically

necessary + suff cond.

$f = F(\phi, \psi)$

necessary:

$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial x}$

$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial y}$

$\frac{\partial f}{\partial z} = \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial z}$

suff

general sol. $f(xyz) = F(\phi, \psi) = 0$

$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$ 3 vari.

linear homog eq. sol.

$f = F(\phi, \psi)$

$P_1\frac{\partial u}{\partial x_1} + P_2\frac{\partial u}{\partial x_2} + \cdots + P_n\frac{\partial u}{\partial x_n} = 0$

$\frac{\partial x_1}{P_1} = \frac{\partial x_2}{P_2} = \cdots = \frac{\partial x_n}{P_n}$ a system of ord. dif. eq.

$\frac{dx_2}{dx_1} = \frac{P_2}{P_1}, \quad \frac{dx_3}{dx_1} = \frac{P_3}{P_1}, \cdots \frac{dx_n}{dx_1} = \frac{P_n}{P_1}$

$x_2 = f_2(x_1; \alpha_1, \alpha_2 \ldots \alpha_{n-1})$

- - -

$x_n = f_n(x_1; \alpha_1, \alpha_2, \cdots \alpha_{n-1})$

$\therefore\ \alpha_1 = \varphi_1(x_1, x_2 \cdots x_n)$

$\alpha_2 = \varphi_2(x_1, x_2 \cdots x_n)$

— —

$\alpha_{n-1} = \varphi_{n-1}(x_1, x_2 \cdots x_n)$

general sol. $u = F(\varphi_1, \varphi_2 \cdots \varphi_{n-1})$

$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = dt$, t new parameter

$\frac{dx_1}{dt} = P_1(x_1, x_2 \cdots x_n)$

$\frac{dx_2}{dt} = P_2(x_1, x_2 \cdots x_n)$ stationary mot. eq. P. t = indep.

— —

$\frac{dx_n}{dt} = P_n(x_1, x_2 \cdots x_n)$

$$P_1\frac{\partial u}{\partial x_1} + P_2\frac{\partial u}{\partial x_2} + \cdots + P_n\frac{\partial u}{\partial x_n} = P_0(x_1, x_2 \cdots x_n)$$

nonhomogeneous sol.

$f(x_1, x_2 \cdots x_n, u) = 0$ integral surface

$$\begin{cases} \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x_1} = 0 \\ \quad - \\ \frac{\partial f}{\partial x_n} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x_n} = 0 \end{cases}$$

$$P_1\frac{\partial f}{\partial x_1} + P_2\frac{\partial f}{\partial x_2} + \cdots + P_n\frac{\partial f}{\partial x_n} + P_0\frac{\partial f}{\partial u} = 0$$

homogeneous eq.

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{du}{P_0}$$

Ex. $(bz - cy)p + (cx - az)q = ay - bx$

$$\frac{dx}{bz-cy} = \frac{dy}{cx-az} = \frac{dz}{ay-bx} = dt$$

$dx = (bz - cy)\,dt$

$dy = (cx - az)\,dt$

$dz = (ay - bx)\,dt$

a	x
b	y
c	z

$a\,dx + b\,dy + c\,dz = 0$

$ax + by + cz = \alpha$

$x\,dx + y\,dy + z\,dz = 0$

$x^2 + y^2 + z^2 = \beta$

$F(x^2+y^2+z^2, ax+by+cz) = 0$ general sol.

or $x^2+y^2+z^2 = \phi(ax+by+cz)$ surface of revolution

surface

$ax + by + cz = k$ plane

$x^2+y^2+z^2 = \phi(k)$ sphere

intersect circle

axis of parallel planes

circle circle centre

Z axis axis surface of revol.

$a, b = 0,\ c = 1 \quad xq - yp = 0$

Ex. $xp + yq = 0$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$$

$$\log x = c + \log y \qquad \frac{y}{x} = \alpha \qquad z = \beta$$

$$z = \varphi\left(\frac{y}{x}\right)$$ general sol.

or $\frac{y}{x} = \psi(z)$ Conoid

Langrange - Charpit's method linear [illegible]

$F(x, y, z, p, q) = 0$ [illegible] order general

∞^5 (x, y, z, p, q) [illegible] surface elements [illegible] case.

∞^4 [illegible] surface [illegible]

$\phi(x, y, z, p, q) = \alpha$ α, arbitrary const.

[illegible] introduce [illegible] satisfy [illegible] surface element [illegible]

$F = 0$ $\phi = \alpha$ [illegible] combine [illegible] ∞^3 [illegible] surf el [illegible]

[illegible] p, q [illegible] solve [illegible]

$$p = \varphi(x, y, z, \alpha)$$

$$q = \psi(x, y, z, \alpha)$$

[illegible] p, q [illegible] surface [illegible] element [illegible]

[illegible] surface [illegible]

$$z = \theta(x, y)$$

$$\frac{\partial \theta}{\partial x} = p, \quad \frac{\partial \theta}{\partial y} = q$$ [illegible]

[illegible] $$\frac{\partial^2 \theta}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$ [illegible]

[illegible] 満足 [illegible] $\phi = \alpha$ [illegible]

$p = \varphi$, $q = \psi$ [illegible] integration [illegible] solution [illegible]

即 $dz = p\,dx + q\,dy$ [illegible] integration [illegible]

$z = \theta(x, y)$ [illegible]

[illegible] $$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x}$$ [illegible] $$dz = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy$$ [illegible]

$z = \theta(x, y)$ [illegible]

然る[illegible]

$F(x\, y\, z\, p\, q) = 0$

$\phi(x\, y\, z\, p, q) = \alpha$ [illegible] $x\, y$ [illegible] diff [illegible]

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad (3)$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad (4)$$

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$ [illegible]

(1) (3) [illegible] $\frac{\partial p}{\partial x}$ [illegible] eliminate [illegible]

(2) (4) [illegible] $\frac{\partial q}{\partial y}$ [illegible]

[illegible] $\frac{\partial q}{\partial x} = (\quad)$, $\frac{\partial p}{\partial y} = (\quad)$ [illegible] equate [illegible]

故に [illegible] eq. [illegible] $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ [illegible] eliminate [illegible]

$\frac{\partial p}{\partial x}, \frac{\partial q}{\partial y}, \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ [illegible]

$$\begin{vmatrix} \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} & \frac{\partial F}{\partial p} & \frac{\partial F}{\partial q} & 0 \\ \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} & 0 & \frac{\partial F}{\partial p} & \frac{\partial F}{\partial q} \\ \frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial p} & \frac{\partial \phi}{\partial q} & 0 \\ \frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} & 0 & \frac{\partial \phi}{\partial p} & \frac{\partial \phi}{\partial q} \end{vmatrix} = 0$$

故に、partial dif. eq. にて ϕ の solution にして

$\frac{\partial(F\phi)}{\partial(p.q)} \neq 0$ 　$=0$ ならば ϕ は F と $p.q$ に関して func. なり

$$\frac{\partial F}{\partial p}\frac{\partial \phi}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial \phi}{\partial y} + \left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\right)\frac{\partial \phi}{\partial z} - \left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)\frac{\partial \phi}{\partial p} - \left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0$$

$\phi(x\,y\,z\,p\,q) = 0$ homogeneous linear part. dif. eq. なり

$$\frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial q}} = \frac{dz}{p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}} = \frac{dp}{-\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)} = \frac{dq}{-\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)} \quad (I)$$

$p.q$ に関する

ϕ の general sol. を求むるには ~~arbitrary const~~ を含む

sol. 中の一つを定むるには α と置くべし

$\phi(x\,y\,z\,p\,q) = \alpha$

$F(x\,y\,z\,p\,q) = 0$

より $p = \varphi(x\,y\,z\,\alpha)$

$q = \psi(x\,y\,z\,\alpha)$ を求む

$dz = p\,dx + q\,dy$ を integrate すれば 更に一つの β

なる arbitrary const. を含む

$z = \chi(x, y, z, \alpha, \beta)$ 　　complete solut. なり

之を Lagrange-Charpit's Method といふ I の formula

は時々之を用ふること多し

Ex. 1. 　$F(p\,q) = 0$

$\frac{dp}{0} = \frac{dq}{0} =$

$p = \alpha$ 　or $q = \alpha$

$p=\alpha, q=\beta$ 　$\phi \equiv p = \alpha$ 　$F(p.q) = 0$

$dz = p\,dx + q\,dy$ より

$dz = \alpha\,dx + \beta\,dy$

$z = \alpha x + \beta y + \gamma$ は plane, α, γ arbitrary const.

α, β は $F(\alpha, \beta) = 0$ なる 関係を有す

Ex. 2. 　$F(x, p\,q) = 0$

$\frac{dq}{0} =$ 　　　　$q = \alpha$

$p = \varphi(x, \alpha)$

$dz = \varphi(x, \alpha)dx + \alpha\,dy$

$z = \int \varphi(x,\alpha)\,dx + \alpha y + \beta$

Ex. 3. 　$F(y.p.q) = 0$ 　　　$p = \alpha$

Ex. 4 　$F(z.p.q) = 0$

$\frac{dp}{p} = \frac{dq}{q} =$

$\frac{q}{p} = \alpha$

$F(z.p.\alpha p) = 0$ 　$p = \varphi(z, \alpha)$ 　$q = \alpha\varphi(z.\alpha)$

$\frac{dz}{\varphi(z,\alpha)} = dx + \alpha\,dy$

$\int \frac{dz}{\varphi(z,\alpha)} = x + \alpha y + \beta$

Ex. 5. $F \equiv f(x.p) - \varphi(y.q) = 0$

$\frac{\partial F}{\partial z} = 0$ 　$\frac{\partial F}{\partial x} = \frac{\partial f(x,p)}{\partial x}$

$\frac{\partial F}{\partial p} = \frac{\partial f(x,p)}{\partial p}$

$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial p} dp = 0$

$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial p} dp = 0 = df$

$f(x, p) = \alpha$

$\varphi(y, q) = \alpha$

$p = \phi(x, \alpha)$

$q = \psi(y, \alpha)$

$dz = p\,dx + q\,dy = \phi(x,\alpha)\,dx + \psi(y,\alpha)\,dy$

$z = \int \phi(x,\alpha)\,dx + \int \psi(y,\alpha)\,dy + \beta$ complete sol

Ex. 6. Generalized Clairaut's equation

$z = px + qy + f(p, q)$ plane — Clairaut's eq. $y = px + f(p)$

$F \equiv z - px - qy - f(p, q)$

$\frac{dp}{0} = \frac{dq}{0}$

$p = \alpha \quad q = \beta$

$F = 0$

$\phi = \alpha$

$\psi = \beta$ [illegible] surface [illegible]

$z = \alpha x + \beta y + f(\alpha, \beta)$ complete sol plane

α, β [illegible] plane [illegible] envelope [illegible]

$\begin{cases} x + \frac{\partial f}{\partial \alpha} = 0 \\ y + \frac{\partial f}{\partial \beta} = 0 \end{cases}$

$z = \alpha x + \beta y + f(\alpha, \beta)$ [illegible] α, β [illegible] eliminate [illegible]

envelope [illegible] singular sol [illegible]

particular case [illegible]

$z - px - qy = r\sqrt{1 + p^2 + q^2}$

$\frac{z - px - qy}{\sqrt{1 + p^2 + q^2}} = r$

Origin [illegible] surface element [illegible] [illegible] const. [illegible]

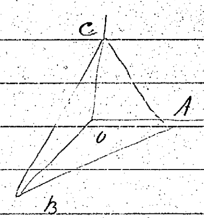
[illegible] Clairaut's eq. [illegible] sphere [illegible] tangential plane [illegible] geometry [illegible] sphere [illegible] singl. sol [illegible]

[illegible] surface [illegible] (x, y, z) [illegible] tangential plane [illegible]

$(\zeta - z) - p(\xi - x) - q(\eta - y) = 0$

[illegible] x axis [illegible] intercept

$y = \zeta = 0 \quad \xi = \frac{-(z - px - qy)}{p}$



y axis [illegible] intercept $\eta = \frac{z - px - qy}{q}$

z axis " $\zeta = z - px - qy$

$OA + OB + OC = (z - px - qy)\left(1 - \frac{1}{p} - \frac{1}{q}\right) = C$

$z - px - qy = \frac{C}{1 - \frac{1}{p} - \frac{1}{q}}$ Clairaut's eq

sol $z = \alpha x + \beta y + \frac{C}{1 - \frac{1}{\alpha} - \frac{1}{\beta}}$ [illegible] envelope [illegible] surface

$OA \cdot OB \cdot OC = \frac{(z - px - qy)^3}{pq} = C$

之ト[illegible] Lagrange-Charpit's method ヲ apply スルコト[illegible]

Legendre's transformation

(x, y, z, p, q) $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ ヲ

$\left(X, Y, Z, P = \frac{\partial Z}{\partial X}, Q = \frac{\partial Z}{\partial Y}\right)$ ニ transform ス

$$(1)\begin{cases} Z = px + qy - z \\ X = p \\ Y = q \end{cases}$$

$$\begin{cases} dz = p\,dx + q\,dy \\ dZ = P\,dX + Q\,dY \end{cases}$$

$$dZ = P\,dX + Q\,dY = P\,dp + Q\,dq$$
$$= x\,dp + y\,dq + \underbrace{p\,dx + q\,dy - dz}_{0}$$

$$\therefore P = x, \quad Q = y$$

$$(2)\begin{cases} z = PX + QY - Z \\ x = P \\ y = Q \end{cases}$$

(1) ト (2) トハ [illegible] reciprocal ナル transformation ニシテ Legendre's transf. ナリ

[illegible] eq. ハ

$F(x, y, z, p, q) = 0$ [illegible] (1)

$F(P, Q, PX + QY - Z, X, Y) = 0$ [illegible] (2)

ノ sol. $Z = \phi(X, Y)$ トスレバ (2ノsol.)

$$z = X\frac{\partial \phi}{\partial X} + Y\frac{\partial \phi}{\partial Y} - \phi(X, Y)$$
$$x = \frac{\partial \phi}{\partial X}, \quad y = \frac{\partial \phi}{\partial Y}$$

ヨリ X, Y ヲ elimination スレバ 元ノ eq. ノ sol. [illegible] (1ノsol.)

即 X, Y ヲ parameter トシタ surface ノ eq. [illegible]

此ハ (1) ハ p, q [illegible] x y z [illegible] transformation ニ [illegible] 此 transformation ハ point coord. ト plane coord. ト [illegible]

此ハ [illegible]

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

$$\begin{cases} dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy \\ dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s\,dx + t\,dy \\ dP = R\,dX + S\,dY \\ dQ = S\,dX + T\,dY \end{cases}$$

$X = p$, $Y = q$, $x = P$, $y = Q$ [illegible]

$$dx = R\,dp + S\,dq = dx(\overline{rR + sS}^{\,1}) + dy(\overline{sR + tS}^{\,0})$$
$$dy = S\,dp + T\,dq = dx(\underbrace{rS + sT}_{0}) + dy(\underbrace{sS + tT}_{1})$$

此ハ identity ナレバ

$$r = \frac{T}{RT - S^2}, \quad s = \frac{-S}{RT - S^2}, \quad t = \frac{R}{RT - S^2}$$

$$rt - s^2 = \frac{1}{RT - S^2}$$

又 $R = \frac{t}{rt - s^2}$, $S = \frac{-s}{rt - s^2}$, $T = \frac{r}{rt - s^2}$

$$RT - S^2 = \frac{1}{rt - s^2}$$

r, s, t と R, S, T の [illegible] は reciprocal なり

此 transf. は second order の dif eq. を 2nd order の dif eq. にする

$$(1+q^2)r - 2pqs + (1+p^2)t = 0 \quad \text{minimal surface の}$$

の solution を transform すると eq.

$$(1+Y^2)T + 2XYS + (1+X^2)R = 0$$ となり 同形なり

Linear partial dif eq of the 2nd order

2nd derivative に linear なる

$$A(x\,y\,z\,p\,q)\frac{\partial^2 z}{\partial x^2} + 2B(xyzpq)\frac{\partial^2 z}{\partial x\partial y} + C(xyzpq)\frac{\partial^2 z}{\partial y^2} = D(xyzpq)$$

coefficient は $x\ y\ z\ p\ q$ の f

其 special case

(1) $$A(xyz)\frac{\partial^2 z}{\partial x^2} + 2B(xyz)\frac{\partial^2 z}{\partial x\partial y} + C(xyz)\frac{\partial^2 z}{\partial y^2} = D(xyzpq)$$

[illegible]

transform of variable にて simplify する

$\xi = \xi(x\,y)$, $\eta = \eta(x\,y)$ ξ, η は [illegible]

(1) の 同等 [illegible]

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$A\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial \xi\partial \eta}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}$$

$$2B\ \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial^2 z}{\partial \xi\partial \eta}\left(\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial x}\right)\right) + \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial x\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x\partial y}$$

$$C\ \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial \xi\partial \eta}\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) + \frac{\partial^2 z}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}$$

$$A'(\xi\,\eta\,z)\frac{\partial^2 z}{\partial \xi^2} + 2B'(\xi\,\eta\,z)\frac{\partial^2 z}{\partial \xi\partial \eta} + C'(\xi\,\eta\,z)\frac{\partial^2 z}{\partial \eta^2} = D'\left(\xi\,\eta\,z\,\frac{\partial z}{\partial \xi}\,\frac{\partial z}{\partial \eta}\right)$$

$$A' = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2$$

$$C' = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + C\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$B' = A\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + B\left\{\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial x}\right)\right\} + C\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

A', B', C' は 同等 [illegible] $\xi\ \eta$ は [illegible]

$A' = C' = 0$ [illegible]

[illegible] $$A\left(\frac{\partial u}{\partial x}\right)^2 + 2B\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + C\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

の sol. として $\xi\ \eta$ [illegible] independent

ノ根ヲ[illegible] 此..

$$A\left(\frac{\alpha}{\beta}\right)^2 + 2B\left(\frac{\alpha}{\beta}\right) + C = 0$$

$$\therefore \frac{\alpha}{\beta} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \qquad \frac{\partial u}{\partial x} = \frac{-B \pm \sqrt{B^2 - \cancel{A}C}}{A}\frac{\partial u}{\partial y}$$

$B^2 - AC \neq 0$ ナラバ 2ツノ根アリ

$B^2 - AC > 0$ ナラバ real sol., real transformation

$B^2 - AC < 0$ imag. sol. imag. transform ニヨリ

結局.. $$\frac{\partial^2 z}{\partial \xi \partial \eta} = M\left(\xi\, \eta\, z\, \frac{\partial z}{\partial \xi}\, \frac{\partial z}{\partial \eta}\right) \qquad (2)$$

ξ, η ハ imaginary func. トナリ且ツ conjugate トナル

$B^2 - AC = 0$ ξ ハ[illegible] η ハ[illegible]

他ノ方程式ニヨリ

η : $C' = 0$

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = 0 \qquad A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y} = 0$$

ナルヲ以テ $AC = B^2$ ナルヲ以テ $B\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial y} = 0$

$$B' = \left(\frac{\partial \xi}{\partial x}\right)\left[A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right\} + \left(\frac{\partial \xi}{\partial y}\right)\left(B\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial y}\right) = 0$$

ξ ハ任意ナル

$$\frac{\partial^2 z}{\partial \xi^2} = M\left(\xi\, \eta\, z\, \frac{\partial z}{\partial \xi}\, \frac{\partial z}{\partial \eta}\right) \qquad (3)$$

(2), (3) ハ standard form ナリ

(2) ハ $\xi = X + Y,\ \eta = X - Y$ ニテ transform シ

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial X^2} - \frac{\partial^2 z}{\partial Y^2}\right)$$

$$\frac{\partial^2 z}{\partial X^2} - \frac{\partial^2 z}{\partial Y^2} = E\left(X\, Y\, z\, \frac{\partial z}{\partial X}\, \frac{\partial z}{\partial Y}\right) \qquad (4) \text{ real}$$

$AC - B^2 > 0$ ノ時ハ $\xi + i\eta = X,\ \xi - i\eta = Y$

$$\frac{\partial^2 z}{\partial X^2} + \frac{\partial^2 z}{\partial Y^2} = E\left(X\, Y\, z\, \frac{\partial z}{\partial X}\, \frac{\partial z}{\partial Y}\right) \qquad (5) \text{ real.}$$

conduct. of heat
vibrat. of a membra.

(4) ハ hyperbolic type $AC - B^2 < 0$ ナル、Laplace's

(5) ハ elliptic type $AC - B^2 > 0$ equat. potential

(3) ハ ~~[illegible]~~ parabolic type $AC - B^2 = 0$

Ex. Laplace's eq. of potential

ポテンシャル

$$\nabla^2 z \equiv \Delta z \equiv \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{simplest ellipt. type}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \qquad \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0 \qquad \xi = x + iy$$

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0 \qquad \eta = x - iy$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = 0 \qquad z = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \text{ arbitrary f.}$$

$z = \varphi(x + iy) + \psi(x - iy)$ general sol. required

real sol. ハ i ノ sign ヲ[illegible] real ナリ

$$z = \varphi(x + iy) + \psi(x - iy) = \varphi(x - iy) + \psi(x + iy)$$

$$z = \frac{1}{2}\left\{\varphi(x + iy) + \varphi(x - iy)\right\} + \frac{1}{2}\left\{\psi(x + iy) + \psi(x - iy)\right\}$$

$$\varphi(x + iy) = U(xy) + iV(xy)$$

$$\varphi(x - iy) = U(xy) - iV(xy)$$

$$\frac{1}{2}\left\{\varphi(x + iy) + \varphi(x - iy)\right\} = U(xy)$$

z [illegible] $\varphi + \psi$ real part [illegible] $x+iy$ arbitrary f [illegible] real part [illegible] real part of $F(x+iy)$

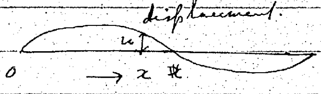
$-iF(x+iy) = -i\Phi + \Psi$ $\qquad F = \phi + i\psi$

[illegible] $\Delta z = 0$ [illegible] real sol. [illegible]

$x+iy$ [illegible] f [illegible] real part [illegible] imaginary part [illegible] complex variable funct. [illegible] real & imaginary part [illegible] Laplace's eq. [illegible] sol. [illegible] funct. theory [illegible]

Ex. 2. $\quad a^2 \dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0 \qquad a$ const. simplest hyp. type

[illegible] $\quad a^2 \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0$



$B^2 - AC = a^2 > 0$ $\quad$ hyp. type $\qquad$ string vibrat.

$a^2\left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2 = 0$

$a\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \qquad \dfrac{dx}{a} = \dfrac{dy}{1} \qquad \begin{array}{l} \xi = x - ay \\ \eta = x + ay \end{array}$

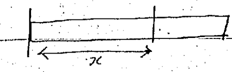
$a\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

$\dfrac{\partial^2 z}{\partial \xi \partial \eta} = 0 \qquad z = \varphi(\xi) + \psi(\eta)$

$\qquad\qquad = \varphi(x - ay) + \psi(x + ay)$

Ex. 3 $\quad \dfrac{\partial^2 z}{\partial x^2} = k \dfrac{\partial z}{\partial y}$ $\quad$ parabolic type.

z [illegible] temp. t [illegible] conduct. of heat



[illegible] boundary condit. [illegible] initial cond. [illegible] general sol. [illegible] arbitrary f [illegible] moment [illegible] state [illegible]

Orthogonal system of functions

$1, \cos x, \cos 2x, \cos 3x, \dots \cos nx, \dots$

$\sin x, \sin 2x, \sin 3x, \dots \sin nx, \dots$

[illegible] system [illegible]

$\displaystyle\int_0^{2\pi} \sin mx \sin nx \, dx = 0 \qquad m \neq n \qquad \cos(m+n) - \cos(m-n)$

$\displaystyle\int_0^{2\pi} \cos mx \cos nx \, dx = 0 \qquad m \neq n$

$\displaystyle\int_0^{2\pi} \sin mx \cos nx \, dx = 0 \qquad m \gtreqless n$

$\displaystyle\int_0^{2\pi} (\sin nx)^2 dx = \int_0^{2\pi} (\cos mx)^2 dx = \pi$

$\displaystyle\int_0^{2\pi} dx = 2\pi \qquad m = 0.$

[illegible] infinite no. [illegible] system of f [illegible] integrati [illegible] 0 [illegible]

$\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots \varphi_n(x), \dots$

$\displaystyle\int_a^b \varphi_m(x)\varphi_n(x)\, dx = 0 \qquad m \gtrless 0$

[illegible] orthogonal system [illegible] orthogonal system [illegible] (orthogonal [illegible] direct [illegible] sum [illegible] infinite dimension [illegible] space [illegible]) orthog. syst. [illegible]

Legendre's polynomials [illegible]

$P_0(x),\ P_1(x),\ P_2(x),\ \ldots,\ P_n(x),\ \ldots$

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0 \qquad m \gtrless n$$

$$\int_{-1}^{+1} \left(P_n(x)\right)^2 dx = \frac{1}{2n+1}$$

Bessel's f.

$J_n(x) = 0$ [illegible] roots $\lambda_1, \lambda_2, \lambda_3, \ldots$

$\sqrt{x}\,J_n(\lambda_1 x),\ \sqrt{x}\,J_n(\lambda_2 x),\ \ldots,\ \sqrt{x}\,J_n(\lambda_m x),\ \ldots$

$$\int_0^1 \left(\sqrt{x}\,J_n(\lambda_i x)\right)\left(\sqrt{x}\,J_n(\lambda_k x)\right) dx = 0 \qquad i \gtrless k$$

[illegible] f. [illegible] orthogonal syst. [illegible]

Legendre's polynomial [illegible] dif. eq.

P_n: $\dfrac{d}{dx}\left((1-x^2)P_n'(x)\right) + n(n+1)P_n(x) = 0$

Bessel's f. [illegible]

$J_n(x)$: $x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0$

$y'' + p(x)\,y' + q(x)\,y = 0$ [illegible] dif. eq. [illegible]

$\dfrac{d}{dx}\left(a(x)\,y'\right) + b(x)\,y = 0$ [illegible]

[illegible] $\dfrac{a'(x)}{a(x)} = p(x) \qquad \dfrac{b(x)}{a(x)} = q(x)$

$a(x) = e^{\int p(x)dx} \qquad b(x) = q\,e^{\int p(x)dx}$ [illegible]

[illegible] J_n [illegible]

$p(x) = \dfrac{1}{x}, \quad q = \dfrac{x^2 - n^2}{x^2}$ [illegible]

$a(x) = x \qquad b(x) = \dfrac{x^2 - n^2}{x}$

$$\frac{d}{dx}\left(x\,J_n'(x)\right) + \frac{x^2 - n^2}{x} J_n(x) = 0$$

$\varphi_m(x) = \sqrt{x}\,J_n(\lambda_m x)$ [illegible] eq.

$J_n(\lambda_m x)$: $\dfrac{d}{\cancel{\lambda_m}dx}\left(x\,J_n'(\lambda_m x)\right) + \dfrac{\lambda_m^2 x^2 - n^2}{x} J_n(\lambda_m x) = 0$

$$\frac{d}{dx}\left(\cancel{\sqrt{x}}\,\varphi_m(x)\right) + \frac{\lambda_m^2 x^2 - n^2}{x}\,\frac{\varphi_m(x)}{\sqrt{x}} = 0$$

[illegible] $P_m(x)$, φ_m [illegible]

$\dfrac{d}{dx}\left(a(x)\,y_m'\right) + b(x, m)\,y_m = 0$ [illegible] $\quad y_\mu$

m [illegible] function [illegible]

[illegible] $P_\mu(x)$, $\varphi_\mu(x)$ [illegible] satisfy [illegible] eq.

$\dfrac{d}{dx}\left(a(x)\,y_\mu'\right) + b(x, \mu)\,y_\mu = 0 \quad y_m$

[illegible]

$$\frac{d}{dx}\left\{y_\mu\left(a(x)\,y_m'\right) - y_m\left(a(x)\,y_\mu'\right)\right\} + \left[b(x, m) - b(x, \mu)\right] y_m y_\mu = 0$$

$$\frac{d^2\left(\sqrt{x}\,J_n(x)\right)}{dx^2} + \left(1 - \frac{4n^2-1}{4x^2}\right)\sqrt{x}\,J_n = 0$$

$$\frac{d^2\left(\sqrt{x}\,J_n(\lambda_m x)\right)}{dx^2} + \left(\lambda_n^2 - \frac{4n^2-1}{4x^2}\right)\sqrt{x}\,J_n(\lambda_m x) = 0$$

$$\int_\alpha^\beta (b(x,\mu)-b(x,m))\,\varphi_m\varphi_\mu\,dx = \Big\{a(x)(\varphi_\mu\varphi_m' - \varphi_\mu'\varphi_m)\Big\}_\alpha^\beta$$

右辺 0 ならば 左辺 integral 0 となる [illegible]

P_m では $b(x,m) = m(m+1)$ とすれば

$$\{m(m+1)-\mu(\mu+1)\}\int_{-1}^{+1} P_m(x)P_\mu(x)\,dx = \Big\{(1-x^2)(P_\mu P_m' - P_\mu' P_m)\Big\}_{-1}^{+1} = 0$$

$m \neq \mu$ ならば $\int_{-1}^{+1} P_m(x)P_\mu(x)\,dx = 0$

φ_m では $\frac{\lambda^2 m x^2 - n^2}{x\sqrt{x}} = b(x,m)$

$$(\lambda_\mu^2 - \lambda_m^2)\int_0^1 \sqrt{x}\,\varphi_m(x)\varphi_\mu(x)\,dx = \Big\{\sqrt{x}(\varphi_\mu\varphi_m' - \varphi_\mu'\varphi_m)\Big\}_0^1 = 0$$

$$\varphi_\mu(1) = [\sqrt{x}\,J_n(\lambda_\mu x)]_{x=1} = J_n(\lambda_\mu) = 0 \quad \lambda \text{ root}$$

$$(\lambda_m - \lambda_\mu)\int_0^1 \varphi_m(x)\varphi_\mu(x)\,dx + \Big\{\varphi_\mu(x)\frac{d}{dx}\varphi_m(x) - \varphi_m(x)\frac{d}{dx}\varphi_\mu(x)\Big\}_0^1 = 0$$

$$\int_0^1 \varphi_m(x)\varphi_\mu(x)\,dx = 0$$

一般: $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ が orthogonal syst. とは $\int_\alpha^\beta \varphi_m(x)\varphi_n(x)\,dx = 0$, $m \gtrless n$ なるとき

一つの函数 x の f が $\varphi_1, \varphi_2, \dots$ で expansible であるとき

$$f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \dots + a_n\varphi_n(x) + \dots \quad \text{(uniform conv.)}$$

であるとき $a_1, a_2, \dots$ を求める

$\int_\alpha^\beta \varphi_m(x)\varphi_n(x)\,dx$ [illegible]

$$\int_\alpha^\beta \varphi_n(x) f(x)\,dx = a_n \int_\alpha^\beta (\varphi_n(x))^2\,dx$$

uniform conv. なら term by term integ. [illegible]

$$a_n = \frac{\int_\alpha^\beta f(x)\varphi_n(x)\,dx}{\int_\alpha^\beta \varphi_n^2(x)\,dx}$$

此の例は trigonometric f. の series である f. は periodic f.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$$

とすれば $a_n = \frac{\int_0^{2\pi} f(x)\cos nx\,dx}{\int_0^{2\pi}\cos^2 nx\,dx} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$,

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$, $\frac{a_0}{2} = \int_0^{2\pi} f(x)\,dx$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

此の Fourier の conduct of heat の研究に用いた trigon. series を Fourier's series といい a_n, b_n を Fourier's coeff. という

此の orthogon. syst. の special case である

$a_n = \frac{\int_\alpha^\beta f(x)\varphi_n(x)\,dx}{\int_\alpha^\beta \varphi_n^2(x)\,dx}$ を general F. coeff. という

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2\cos 2x + b_2\sin 2x) + \dots + (a_n\cos nx + b_n \sin nx) + \dots$$

の ~~Fourier~~ expansible で uniformly conv. なら

次のとおり $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

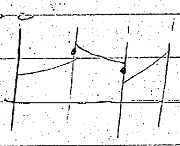
$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$ となる

此 series ノ convergency ト expansibility ノ condition
ハ Fourier ノ [illegible] [illegible] 此ガ [illegible]
Dirichlet (Paris ニ [illegible] Fourier ニ [illegible]) ニ於テ
[illegible] Dirichlet ノ condition
[illegible]

1. $f(x)$ ガ $(0, 2\pi)$ ニテ finite
2. one-valued, continuous except finite numbers of discontinuous points
3. finite no of maxima & minima
4. if c : a discont. pt, $\lim_{\varepsilon \to 0} f(c+\varepsilon)$ & $\lim_{\varepsilon \to 0} f(c-\varepsilon)$ exist

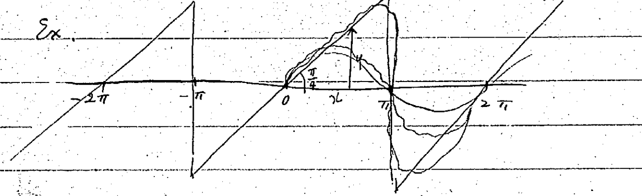
此 [illegible] ~~func~~ f ハ convergent series ニ expansible [illegible] Coeff. [illegible] discont. pt. c ニテ

$$\frac{f(c+0) - f(c-0)}{2}$$ [illegible]



[illegible] applicable
funct. ノ [illegible] variable ノ corresp. ニテ [illegible] Fourier's series [illegible] Dirichlet ノ [illegible]

Ex.



periodic f
discont. at $\pm\pi$, $\pm 3\pi$, ...

此 function ニテ [illegible]

$f(x) = x \qquad 0 \leqq x \leqq \pi$

$\quad = x - 2\pi, \quad \pi \leqq x \leqq 2\pi$

Dirichlet ノ cond. [illegible] expansible

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0 = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(x-2\pi)\cos nx\,dx\right]$$

$$= 0$$

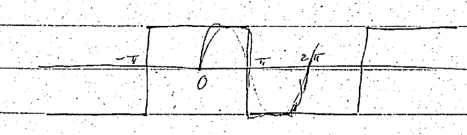
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\,dx + \int_{\pi}^{2\pi}(x-2\pi)\sin nx\,dx\right]$$

$$= -\frac{2\cos n\pi}{n}$$

$$f(x) = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots (-1)^{n-1}\frac{\sin nx}{n} + \cdots\right\}$$

$x = \pi$ ニテ $f(x)$ ハ 0 ニテ [illegible]

Ex.



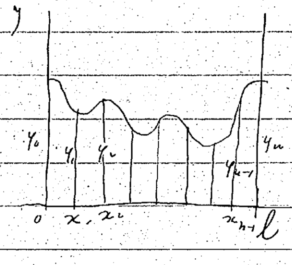
$f(x) = 1 \qquad 0 \leqq x \leqq \pi$

$\quad = -1 \qquad \pi \leqq x \leqq 2\pi$

Fourier's series ハ wave ノ analysis ニ [illegible]
ト period ノ [illegible] f ヲ
Fourier's series ニ expand [illegible] term [illegible]

[illegible] period [illegible] $f(x)$ [illegible]
graph [illegible] $a_0, a_1 \ldots$ [illegible] coeff. [illegible] approximation [illegible] a_0 [illegible] $(0, 2\pi)$ [illegible] area [illegible]
[illegible]

[illegible] periodic curve [illegible] period l.



$y = f(x)$

$\frac{2\pi x}{l} = \theta$

$x = l \quad \theta = 2\pi$

$y = f(x) = f\left(\frac{l\theta}{2\pi}\right) = F(\theta)$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(\theta)\,d\theta$

$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi}\int_0^{2\pi} F(\theta) \begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix} d\theta$

$\frac{2\pi}{l}dx = d\theta \qquad a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{2}{l}\int_0^l f(x) \begin{matrix} \cos \frac{2n\pi}{l}x \\ \sin \frac{2n\pi}{l}x \end{matrix} dx$

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad \int_0^l f(x)\,dx$: area

$\frac{a_0}{2} = \frac{y_0 + y_1 + y_2 + \cdots + y_{n-1}}{n}$

$\frac{a_n}{2} = \frac{y_0 \cos\frac{2n\pi x_0}{l} + y_1 \cos\frac{2n\pi x_1}{l} + \cdots + y_{n-1}\cos\frac{2n\pi x_{n-1}}{l}}{n} \quad y = f(x)\cos\frac{2n\pi x}{l}$

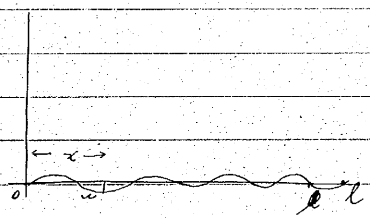
$\frac{b_n}{2} = \frac{y_0 \sin\frac{2n\pi x_0}{l} + y_1 \sin\frac{2n\pi x_1}{l} + \cdots + y_{n-1}\sin\frac{2n\pi x_{n-1}}{l}}{n}$

sin, cos [illegible] x [illegible] equidistance [illegible]
[illegible] (Steinmetz, Engineering Math.)
[illegible] mechanical [illegible] harmonic analyser [illegible]
[illegible]

Legendre's f. Bessel's f. [illegible] expansibility [illegible]
cond. [illegible] Fourier's s. [illegible] Integral
eq. [illegible]

ex. 1. Problem of vibrat. of string

$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ hyperbolic type



u displacement
t time
boundary condit.
$u = 0$ for $x = 0,\ x = l$
initial condit.
$t = 0$ [illegible] $u = f(x)$ (given)

$\frac{\partial u}{\partial t} = \varphi(x)$ for $t = 0$

[illegible] u [illegible]

physical meaning ヲ考ヘル。 x, t ノ variable ト boundary ノ条件ニ関シテ function ヲ求メル問題ヲ Randwertaufgabe (Boundary value problem) ト云フ

$u = e^{\lambda i t} v(x)$ ト置ク (periodic)

$\frac{\partial u}{\partial t} = \lambda i e^{\lambda i t} v(x)$

$\frac{\partial^2 u}{\partial t^2} = -\lambda^2 e^{\lambda i t} v(x)$

$\frac{\partial^2 u}{\partial x^2} = e^{\lambda i t} v''(x)$

$v''(x) + \frac{\lambda^2}{a^2} v(x) = 0$

gen. sol. $v(x) = A \cos\frac{\lambda x}{a} + B \sin\frac{\lambda x}{a}$

u: $\cos\lambda t \sin\frac{\lambda x}{a}$

$\sin\lambda t \sin\frac{\lambda x}{a}$ $\quad \because A\cos\frac{\lambda x}{a} = 0 \quad \because u = 0$ for $x = 0$

$\sin\frac{\lambda l}{a} = 0$ ナラバ $\frac{\lambda l}{a} = n\pi$

$\lambda = \frac{an\pi}{l}$ $\quad n$ integer

$\sum_{n=1}^{\infty} A_n \cos\frac{an\pi}{l} t \sin\frac{n\pi x}{l} + B_n \sin\frac{an\pi}{l} t \sin\frac{n\pi x}{l} = u$

ハ convergent ナレバ eq. ヲ満タシ且ツ boundary condit. ヲ満ス

uniformly conv. ナルトキハ initial cond. ヲ満タス。コレヨリ A_n, B_n ヲ定メルコトガ出来ル

$t = 0$ $\quad u = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$

$\frac{\partial u}{\partial t} = \sum B_n \frac{an\pi}{l} \cos\frac{an\pi}{l} t \sin\frac{n\pi x}{l} + \sum A_n \sin(\ ) \sin(\ )$

$\varphi(x) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{l} \sin\frac{n\pi x}{l}$

コレニヨリ A_n, B_n ヲ定メル。f, φ ガ Fourier's series expansible ナラバ定メ得ル

$\frac{\pi x}{l} = \theta$ $\quad A_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l} dx$

$\frac{an\pi}{l} B_n = \frac{2}{l}\int_0^{l} \varphi(x) \sin\frac{n\pi x}{l} dx$

此問題ハ Bernoulli ノ取扱ツタ classical problem デアル

Ex. 2 Problem of conduction of heat.

$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ $\quad$ parabolic type

boundary condit.: $u = 0$ for $x = 0$

$u = 0$ for $x = l$

initial cond.: $u = f(x)$ for $t = 0$

t time

u temp.

$t \to \infty$ $\quad u \to 0$

put $u = e^{-\lambda^2 t} v(x)$

$\frac{\partial u}{\partial t} = -\lambda^2 e^{-\lambda^2 t} v(x)$

$\frac{\partial^2 u}{\partial x^2} = e^{-\lambda^2 t} v''(x)$

$$v''(x) + \frac{\lambda^2}{a^2} v(x) = 0$$

$$v = A\cos\frac{\lambda}{a}x + B\sin\frac{\lambda}{a}x$$

$v = 0$ for $x = 0$ $\quad A = 0$

$v = 0$ for $x = l$ $\quad \lambda = \frac{an\pi}{l}$

$$u = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{a\pi}{l}\right)^2 n^2 t} \sin\frac{n\pi x}{l} \qquad n = 1, 2, 3 \cdots$$

C_n [illegible] initial condit. [illegible]

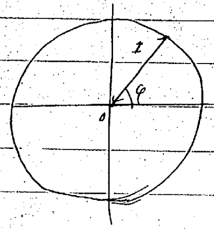
$$f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \quad \text{if expansible}$$

$$C_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

initial cond. [illegible] u [illegible] temp. distribut. [illegible] discontinuous [illegible] Fourier's series expansible [illegible] (Dirichlet's cond.) [illegible]

Ex. 3. Problem of potential in a plane

$$\Delta(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{elliptic type}$$



unit circle [illegible] pot. given [illegible] potential [illegible]

bound. c. $u = f(\varphi)$ for $r = 1$ $\quad (r, \varphi)$ polar coord.

polar coor. [illegible]

$$\Delta(u) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

$u = R(r)\cdot\Phi(\varphi)$ [illegible]

$$R''(r)\phi(\varphi) + \frac{1}{r}R'(r)\phi(\varphi) + \frac{1}{r^2}R(r)\phi''(\varphi) = 0$$

$$\frac{r^2}{R(r)}\left(R''(r) + \frac{1}{r}R'(r)\right) + \frac{\phi''(\varphi)}{\phi(\varphi)} = 0$$

$$\frac{r^2}{R(r)}\left(R''(r) + \frac{1}{r}R'(r)\right) = -\frac{\phi''(\varphi)}{\phi(\varphi)} = \text{const} = \mu^2$$

[illegible] r, φ [illegible] funct. [illegible]

$$r^2 R''(r) + rR'(r) - \mu^2 R(r) = 0 \qquad (1)$$

$$\Phi''(\varphi) + \mu^2\phi(\varphi) = 0 \qquad (2)$$

(2) [illegible] $\phi(\varphi) = A\cos\mu\varphi + B\sin\mu\varphi$

(1) [illegible] $R(r) = r^\rho \quad R'' = \rho(\rho-1)r^{\rho-2} \quad R' = \rho r^{\rho-1}$

$\rho(\rho-1) + \rho - \mu^2 = 0$ or $\rho^2 = \mu^2$ charact. eq.

$R(r) = Cr^{\mu} + Dr^{-\mu}$

[illegible] $r = 0$ [illegible] sol. [illegible]

$D = 0$

μ: posit. integer [illegible]

$$u = \sum_{n=0}^{\infty} r^n (A_n\cos n\varphi + B_n\sin n\varphi)$$

$r = 1$ $\quad \frac{A_0}{2} + \sum_{n=0}^{\infty}(A_n\cos n\varphi + B_n\sin n\varphi) = f(\varphi)$ expansible into F's series

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi\, d\varphi$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi$$

$$B_0 = 0$$

ニ α_n [illegible]

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,\frac{1-r^2}{1-2r\cos(\varphi-\psi)+r^2}\,d\psi$$ ニナル

Poisson's integral トス

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\psi)\,\frac{1-r^2}{\;''\;}\,d\psi$$

ニヨリ u ノ φ, ψ ノ f ノ [illegible] integral eq. of the 1st kind トナル

[illegible]

$$\frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos(n\varphi-\psi)\,d\psi = a_n\cos n\varphi + b_n\sin n\varphi$$

$$u = \frac{1}{2\pi}\left\{\int_0^{2\pi} f(\psi)\,d\psi + \sum_{n=1}^{\infty}\int_0^{2\pi} f(\psi)\,(2\cos n(\varphi-\psi))\,d\psi\right\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\left\{1+\sum_{n=1}^{\infty} r^n(2\cos n(\varphi-\psi))\right\}d\psi$$ 此ヲ入レテ [illegible]

$$1+\sum_{n=1}^{\infty} r^n\left(e^{in(\varphi-\psi)}+e^{-in(\varphi-\psi)}\right)$$

$$= 1+\sum_{n=1}^{\infty}\left(re^{i(\varphi-\psi)}\right)^n + \sum_{n=1}^{\infty}\left(re^{-i(\varphi-\psi)}\right)^n$$ $1+x+x^2+\dots = \frac{1}{1-x}$

$$= 1+\frac{re^{i(\varphi-\psi)}}{1-re^{i(\varphi-\psi)}}+\frac{re^{-i(\varphi-\psi)}}{1-re^{-i(\varphi-\psi)}}$$

$$= \frac{1-re^{i(\varphi-\psi)}-re^{-i(\varphi-\psi)}+r^2+re^{i(\varphi-\psi)}-r^2+re^{-i(\varphi-\psi)}-r^2}{(1-re^{i(\varphi-\psi)})(1-re^{-i(\varphi-\psi)})} = \frac{1-r^2}{\;''\;}$$

$$= \frac{1-r^2}{1-2r\cos(\varphi-\psi)+r^2}$$

$$u = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,\frac{1-r^2}{1-2r\cos(\varphi-\psi)+r^2}\,d\psi$$

今ノ結果ハ連続ナ境界値ニ就テハ上ノ通リ [illegible] 一様
ナ範囲ニ於テ [illegible]

Ex. 4. potential in space

$$\Delta(u) \equiv \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$$ Laplace eq.
Elliptic type

boundary cond. a given surface ニ u given

three variables ノトキ type ヲ決メル

此形ガ $u_1^2+u_2^2+u_3^2$ $\frac{\partial^2 u}{\partial x\partial y} \dots u_1 u_2$

definite quadratic form ノトキ elliptic

indefinite " " hyperbolic

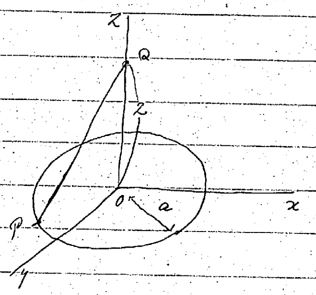
semi-definite " " parabolic

definite トハ 一定ノ sign ($u_1 \dots$ ガ皆0ナラザルトキ)

indef. 時ニ + 時ニ −

semi-def. 常ニ same sign ナルモ 0 トナルコトアリ

一例トシテ $Au_1^2+2Bu_1u_2+Cu_2^2$ $B^2-AC<0$ def.
>0 indef.
$=0$ semi-def.



origin centre [illegible] circular ring [illegible] mass [illegible] distributed [illegible] space [illegible] potential [illegible]

$PQ = \sqrt{a^2+z^2}$

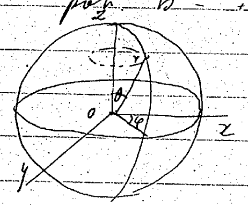
P [illegible] Q [illegible] pol. [illegible] inverse prop

ring [illegible] Q [illegible] pol. $\frac{M}{\sqrt{a^2+z^2}}$ M mass

[illegible] z axis [illegible] sol. [illegible]

mass uniformly distributed [illegible] z [illegible] pol. [illegible]

polar coord. $r\ \theta\ \varphi$ [illegible] φ [illegible] r, θ [illegible] pol. [illegible] φ [illegible] indep.



$$\Delta(u) = r\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{\partial\theta} + \cancel{\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}} = 0$$

φ [illegible] indep. [illegible] sol. [illegible]

$$u = R(r)\,\Theta(\theta)$$

$$r\Theta\frac{d^2(rR(r))}{dr^2} + \frac{R(r)}{\sin\theta}\frac{d(\sin\theta\,\Theta'(\theta))}{d\theta} = 0$$

$$\frac{r}{R(r)}\frac{d^2(rR(r))}{dr^2} = -\frac{1}{\sin\theta\,\Theta}\frac{d(\sin\theta\,\Theta')}{d\theta} = \mu$$

$$\frac{d^2(rR)}{dr^2} - \frac{R}{r}\mu = 0 \;, \quad {}^{(1)}\; \frac{d(\sin\theta\,\Theta')}{d\theta} + \mu\sin\theta\,\Theta = 0$$

(1) [illegible]

$R = r^\rho \quad \rho(\rho+1)r^{\rho-1} - \mu r^{\rho-1} = 0 \quad \rho(\rho+1) = \mu$

(2) [illegible] $\cos\theta = t$ [illegible]

$$\frac{d\Theta}{d\theta} = -\frac{d\Theta}{dt}\sin\theta$$

$$\sin\theta\frac{d\Theta}{d\theta} = -\frac{d\Theta}{dt}\sin^2\theta$$

$$\frac{d}{dt}\left((1-t^2)\frac{d\Theta}{dt}\right) + \mu\Theta = 0$$

$\mu = m(m+1)$ [illegible] Legendre's polynomial [illegible] dif. eq.

$P_m(t) = P_m(\cos\theta)$

[illegible] $\rho = m$ or $-(m+1)$

$R = r^m,\ r^{-(m+1)}$

$\Theta = P_m(\cos\theta),\ Q_m(\cos\theta)$

$r=0$ [illegible] 0 [illegible] pot. [illegible] ∞ [illegible] $r^{-(m+1)}$ [illegible]

Q [illegible] $\cos\theta = 0$ [illegible] ∞ [illegible]

[illegible] sol. [illegible]

$$u = \sum_{m=0}^{\infty} a_m r^m P_m(\cos\theta)$$

boundary cond. [illegible] $\cos\theta = 1$ $\theta = 0$ [illegible] z axis [illegible] $\frac{M}{\sqrt{a^2+z^2}}$

$$u = \frac{M}{\sqrt{a^2+z^2}} = \sum_{m=0}^{\infty} a_m r^m P_m(1) \text{ [illegible] } a_m \text{ [illegible]}$$

$P_m(1) = 1. \quad u = \sum_{m=0}^{\infty} a_m r^m$ boundary cond.

$$\frac{M}{\sqrt{a^2+x^2}} = \frac{M}{a}\left(1+\left(\frac{r}{a}\right)^2\right)^{-\frac{1}{2}}$$ [illegible]

a_n [illegible]

boundary cond. [illegible] $r=1$ [illegible] $u = f(\theta)$ [illegible]

P_n orthogonal syst. [illegible] a_n [illegible]

[illegible] u special form [illegible]

assumption [illegible] sol. [illegible]

unique possible [illegible]

[illegible] Laplace eq. [illegible] sol. [illegible] boundary cond.

[illegible] unit sphere [illegible] $u=1$ [illegible]

$u=0$ [illegible] (cooling [illegible])

$$r\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{\partial\theta} + \cancel{\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\varphi^2}} = 0$$

$u = 1$ for $r=1$ $0 \leq \theta \leq \frac{\pi}{2}$

$u = 0$ for $r=1$ $\frac{\pi}{2} \leq \theta \leq \pi$

$$u = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

boundary cond. [illegible]

$f(\theta) = \sum a_n P_n(\cos\theta)$

$f(\theta) = 1$ $0 \leq \theta \leq \frac{\pi}{2}$

$= 0$ $\frac{\pi}{2} \leq \theta \leq \pi$

or $F(x) = \sum a_n P_n(x)$

$F(x) = 1$ for $0 \leq x \leq 1$

$= 0$ for $-1 \leq x < 0$

$$a_n = \frac{\int_{-1}^{1} F(x) P_n(x)\,dx}{\int_{-1}^{1} P_n^2(x)\,dx \left(= \frac{2}{2n+1}\right)}$$

$$= (2n+1)\int_{-1}^{1} F(x) P_n(x)\,dx = (2n+1)\int_0^1 P_n(x)\,dx$$

[illegible] φ [illegible] independent [illegible] sol. [illegible]

spherical harmonics [illegible]

Spherical harmonics

$$\Delta(u) = r\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0$$

[illegible] $u = R(r)\cdot\Theta(\theta)\cdot\Phi(\varphi)$ [illegible] part. sol. [illegible]

$$r\Theta\Phi\frac{\partial^2(rR)}{\partial r^2} + \frac{1}{\sin\theta}R\Phi\frac{\partial(\sin\theta\,\Theta')}{\partial\theta} + \frac{1}{\sin^2\theta}R\Theta\Phi''(\varphi) = 0$$

$R\Phi\Theta$ [illegible] $\sin^2\theta$ [illegible]

$$\sin^2\theta\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} + \frac{\sin\theta}{\Theta}\frac{\partial(\sin\theta\,\Theta')}{\partial\theta} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}$$

[illegible] θ, r [illegible] φ [illegible] $= \text{const} = n^2$

$$\Phi'' + n^2\Phi = 0 \qquad (1)$$

$$\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} = \frac{n^2}{\sin^2\theta} - \frac{1}{\sin\theta\,\Theta}\frac{\partial(\sin\theta\,\Theta')}{\partial\theta} = \text{const} = m(m+1)$$

[illegible] θ, r [illegible]

$$r\frac{\partial^2(rR)}{\partial r^2} - m(m+1)R = 0 \qquad (2)$$

$$\frac{\partial(\sin\theta\,\Theta')}{\partial\theta} + \sin\theta\,\Theta\left(m(m+1) - \frac{n^2}{\sin^2\theta}\right) = 0$$

$\cos\theta = x$

$$\Theta' = \frac{d\Theta}{d\theta} = -\frac{d\Theta}{dx}\sin\theta$$

$$\sin\theta\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + \sin\theta\,\Theta\left(m(m+1) - \frac{n^2}{1-x^2}\right) = 0$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + \Theta\left(m(m+1) - \frac{n^2}{1-x^2}\right) = 0 \qquad (3)$$

(1) [illegible] $\Phi(\varphi) = A\cos n\varphi + B\sin n\varphi$

(2) [illegible] $R(r) = A'r^m + B'r^{-(m+1)}$

(3) sol. [illegible]

$$\Theta = (1-x^2)^{\frac{n}{2}}\frac{d^nP_m(x)}{dx^n} \quad \text{or} \quad (1-x^2)^{\frac{n}{2}}\frac{d^nQ_m(x)}{dx^n}$$

P, Q … Legendre's polynomial

[illegible]

put $\Theta = (1-x^2)^{\frac{n}{2}}v(x)$

$$\frac{d\Theta}{dx} = (1-x^2)^{\frac{n}{2}}v'(x) - nx(1-x^2)^{\frac{n}{2}-1}v(x)$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) = (1-x^2)^{\frac{n}{2}+1}v''(x) - (n+2)x(1-x^2)^{\frac{n}{2}}v'(x)$$

$$= -nx(1-x^2)^{\frac{n}{2}}v'(x) - nv(x)\left[(1-x^2)^{\frac{n}{2}} - nx^2(1-x^2)^{\frac{n}{2}-1}\right]$$

[illegible] $= -m(m+1)(1-x^2)^{\frac{n}{2}}v(x) + n^2(1-x^2)^{\frac{n}{2}-1}v(x)$

$$(1-x^2)v''(x) - 2(n+1)xv'(x) - (n(n+1) - m(m+1))v(x) = 0$$

Legendre's polynomial [illegible]

$$(1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + m(m+1)V = 0 \qquad V = P_m(x)$$

n回 [illegible]

$$(1-x^2)\frac{d^2(V^{(n)})}{dx^2} + n(-2x)\frac{d(V^{(n)})}{dx} - 2\frac{n(n-1)}{1\cdot 2}V^{(n)}$$
$$-2x\frac{d(V^{(n)})}{dx} - 2nV^{(n)} + m(m+1)V^{(n)} = 0$$

$V^{(n)}(x) = v$

$$(1-x^2)\frac{d^2v}{dx^2} - 2(1+n)x\frac{dv}{dx} + (m(m+1) - n(n+1))v = 0$$

[illegible] eq. [illegible] sol. $\frac{d^nP_m(x)}{dx^n}$, or $\frac{d^nQ_m(x)}{dx^n}$

[illegible]

Φ $\sin n\varphi$, $\cos n\varphi$ linear form

R r^m $r^{-(m+1)}$ [illegible] ″

Θ $(1-x^2)^{\frac{n}{2}}\frac{d^nP_m(x)}{dx^n}$, $(1-x^2)^{\frac{n}{2}}\frac{d^nQ_m(x)}{dx^n}$

m, n any integer [illegible]

$u = R(r)\,\Theta(\theta)\,\Phi(\varphi)$ sol. [illegible]

$r = 0$ [illegible] infinite [illegible] sol. [illegible] $r^{-(m+1)}$ [illegible]

$x = 0$ $(\cos\theta = 0)$ $\frac{d^nQ_m(x)}{dx^n}$ [illegible]

$$x = \cos\theta = 0 \text{ [illegible] } \Theta = \frac{\sin^n\theta\, d^nP_m(\cos\theta)}{d(\cos\theta)^n} = P_m^{(n)}(\cos\theta)$$

$$P_m^{(n)}(\cos\theta) = \frac{(2m)!\sin^n\theta}{2^m m!(m-n)!}\left[(\cos\theta)^{m-n} - \frac{(m-n)(m-n-1)}{2(2m-1)}(\cos\theta)^{m-n-2}\right.$$
$$\left. + \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2\cdot 4(2m-1)(2m-3)}(\cos\theta)^{m-n-4} - \cdots\right]$$

— sol. $r^m(A\cos n\varphi + B\sin n\varphi)P_m^{(n)}(\cos\theta)$

$\cos n\varphi\, P_m^{(n)}(\cos\theta)$

$\sin n\varphi\, P_m^{(n)}(\cos\theta)$ [illegible] Tesseral harmonics

P_m .. m th degree of the m^{th} degree of [illegible]

$n \leq m$ polynomial [illegible] n^{th} order [illegible]

m th deg [illegible] harmonics …

$P_m(\cos\theta),\ \cos\varphi\, P_m^{(1)}(\cos\theta), \dots\ \cos m\varphi\, P_m^{(m)}(\cos\theta)$

$\sin\varphi\, P_m^{(1)}(\cos\theta), \dots\ \sin m\varphi\, P_m^{(m)}(\cos\theta)$

[illegible] $2m+1$

$$Y_m(\theta,\varphi) \equiv A_0 P_m(\cos\theta) + \sum_{n=1}^{m}(A_n\cos n\varphi + B_n\sin n\varphi)P_m^{(n)}(\cos\theta)$$

[illegible] surface spherical harmonics of the m th degree [illegible]

[illegible]

$$u = \sum_{m=0}^{\infty} r^m Y_m(\theta,\varphi) \qquad \text{general sol}$$

$r^m Y_m(\theta,\varphi),\ r^{-(m+1)}Y_m(\theta,\varphi)$ [illegible]

solid spherical harmonics of the m th degree [illegible]

spherical surface [illegible] $r = \text{const.}$ [illegible]

$$u = \sum_{m=0}^{\infty} Y_m(\theta,\varphi)$$

[illegible] $u = F(\theta,\varphi)$ given function [illegible]

$$u = \sum_{m=0}^{\infty} a_m Y_m(\theta\varphi) \quad [illegible]$$

orthogonality [illegible] ~~[illegible]~~

$$\int_0^{2\pi} d\varphi \int_0^{\pi} Y_i(\theta,\varphi)Y_k(\theta,\varphi)\sin\theta\, d\theta = 0 \qquad i \gtrless k$$

[illegible]

$\sin\theta\, d\theta\, d\varphi$ .. surface [illegible] surface element ($d\sigma$)

$\iint Y_i Y_k\, d\sigma$ [illegible]

$U = r^i Y_i(\theta,\varphi) \qquad \Delta(U) = 0$

$V = r^k Y_k(\theta,\varphi) \qquad \Delta(V) = 0$

unit sphere [illegible] vol [illegible] surface integral [illegible]

Green's theorem [illegible]

$$\iiint (U\Delta V - V\Delta U)\, d\omega = \pm\iint \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)$$

[illegible] $= 0$ n normal [illegible] rad. vector [illegible]

$$\frac{\partial U}{\partial n} = \frac{dU}{dr} = i r^{i-1} Y_i(\theta\varphi)$$

$$\frac{\partial V}{\partial n} = \frac{dV}{dr} = k r^{k-1} Y_k(\theta\varphi)$$

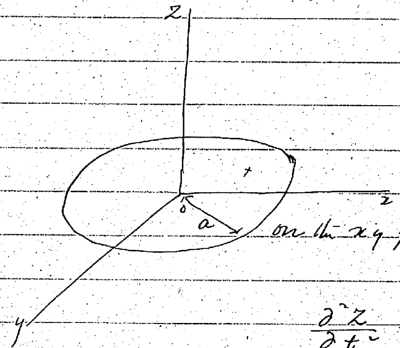
$$k r^i Y_i r^{k-1} Y_k - i r^k Y_k r^{i-1} Y_i = \underset{\neq 0}{(k-i)} \iint \underset{\text{Const}}{r^{i+k-1}} Y_i(\theta\varphi) Y_k(\theta\varphi)\, d\sigma = 0$$

[illegible] $\iint Y_i Y_k\, d\sigma = 0$

[illegible] spherical harmonics [illegible]

Problem of vibration of membranes.
(Diff. eq. of the hyperbolic type)

1.



a circle = elastic membrane [illegible] initial state [illegible] vibration [illegible] pt. z [illegible] displacement [illegible]

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Var. [illegible] hyp. type [illegible]

$f(u_1, u_2, u_3) = u_1^2 - c^2 (u_2^2 + u_3^2)$ [illegible] indefinite form

initial cond. $z = f(x, y)$ for $t = 0$

for simplicity's sake $z = f(r)$ for $t = 0$ [illegible] concentric circle [illegible]

vel. $\frac{\partial z}{\partial t} = 0$ for $t = 0$

boundary cond. $z = 0$ for $r = a$

x, y [illegible] polar coord. [illegible] transform [illegible] cylindrical coord. [illegible] refer [illegible]

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 z}{\partial \varphi^2}}\right)$$ φ [illegible]

$Z = R(r)\,T(t)$ [illegible]

$RT'' = c^2 T\left(R'' + \frac{1}{r}R'\right)$

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = \text{const.} = -m^2$$

$-m^2$ [illegible] $\frac{T''}{cT}$ [illegible] periodic [illegible]

$T'' + c^2 m^2 T = 0$ $\qquad rR'' + R' + m^2 rR = 0$

$T(t) = \cos cmt,\ \sin cmt,$ $\qquad r = \frac{x}{m}$ [illegible]

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0$$

Bessel funct. $J_0(x)$

$= J_0(mr)$

[illegible] $\cancel{Y_0(mr)}$ Bessel f. of 2nd k.

$Z = (A\cos cmt + B\sin cmt)\,J_0(mr)$

$r = a \quad Z = 0 \qquad J_0(ma) = 0 \qquad J_0(x) = 0,\ \mu_1, \mu_2, \mu_3, \ldots$

[illegible] ma [illegible] $J_0 = 0$ [illegible] root [illegible] m [illegible]

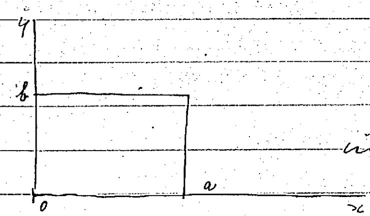
$\frac{\partial z}{\partial t} = 0$ [illegible] $B\sin cmt$ [illegible] term [illegible]
(dif.) [illegible] cos [illegible] $t = 0$ [illegible])

$$Z = \sum_{\nu=0}^{\infty} A_\nu \cos\frac{c\mu_\nu}{a}t\, J_0\left(\frac{\mu_\nu r}{a}\right)$$

$z = f(r)$ for $t = 0$ [illegible] $\sum_{\nu=0}^{\infty} A_\nu J_0\left(\frac{\mu_\nu r}{a}\right) = f(r)$

[illegible] Bessel f. [illegible] orthogonality [illegible] A_ν [illegible]

2.



rectangular membrane

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

init. cond. [illegible] $z = f(x, y)$ for $t = 0$

$\frac{\partial z}{\partial t} = 0$ for $t = 0$

boundary cond: $z=0$ for $x=0,\ x=a$

and for $y=0,\ y=b$

$z = T(t)\, e^{i(\alpha x+\beta y)}$ $= \varphi(x,y)$. 4 special

$T'' e^{i(\alpha x+\beta y)} = c^2 T(-\alpha^2-\beta^2) e^{i(\alpha x+\beta y)}$

$\dfrac{T''}{c^2 T} = -(\alpha^2+\beta^2)$

$T'' + c^2(\alpha^2+\beta^2) T = 0$

$T = A\cos c\sqrt{\alpha^2+\beta^2}\, t + B\sin c\sqrt{\alpha^2+\beta^2}\, t$

$i(\alpha x+\beta y)$: $(\underset{x=0\ z=0}{\cos\alpha x} + i\sin\alpha x)$ $(\cos\beta y + i\sin\beta y)$

$\sin\alpha x \cdot \sin\beta y$

$\sin\alpha a = 0$

$\sin\beta b = 0$

$\alpha a = m\pi$, $\alpha = \dfrac{m\pi}{a}$

$\beta b = n\pi$, $\beta = \dfrac{n\pi}{b}$

$\dfrac{\partial z}{\partial t} = 0$ $B\sin c\sqrt{\alpha^2+\beta^2}$

$z = \sin\dfrac{m\pi}{a}x \sin\dfrac{n\pi}{b}y \cos\left(c\pi\sqrt{\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}}\, t\right)$

$z = \sum_{m,n}^{1,\infty} a_{m,n} \sin\dfrac{m\pi}{a}x \sin\dfrac{n\pi}{b}y \cos\left(c\pi\sqrt{\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}}\, t\right)$

$z = f(xy)$ for $t=0$

$f(xy) = \sum_{m,n}^{1,\infty} a_{mn} \sin\dfrac{m\pi x}{a} \sin\dfrac{n\pi y}{b}$

Fourier's series apply

$\int_0^{2a} f(xy) \sin\dfrac{m\pi x}{a}\,dx = \sum_n^{1,\infty} a_{m,n} \sin\dfrac{n\pi y}{b} \int_0^{2a} \sin^2\dfrac{m\pi x}{a}\,dx$

$\int_0^{2b} \sin\dfrac{n\pi y}{b}\,dy \int_0^{2a} f(xy) \sin\dfrac{m\pi x}{a}\,dx = a_{mn}\left(\int_0^{2a} \sin^2\dfrac{m\pi x}{a}\,dx\right)\left(\int_0^{2b} \sin^2\dfrac{n\pi y}{b}\,dy\right)$

a_{mn}

Fourier's double integral

finite length, conduct finite period, trigonom. series expand infinite period, finite period, infinite, infinite series, integral, Fourier's double integral

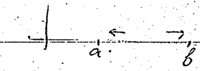
2nd mean value theorem

(a,b) $f(x)$ continuous f.

$\varphi(x)$ monotonous f.

$\int_a^b f(x)\varphi(x)\,dx = \varphi(a+0)\int_a^{\xi_0} f(x)\,dx + \varphi(b-0)\int_{\xi_0}^b f(x)\,dx$

$a < \xi < b$



$$\int_{x_{i-1}}^{x_i} f(x)\,dx = F(x_i) - F(x_{i-1})$$

a x_1 x_2 b ($a = x_0$) [illegible] $F(x) = \int f(x)\,dx$

by ~~Mean value theorem~~ $\int_{x_{i-1}}^{x_i} f\,dx = (x_i - x_{i-1}) f(\xi_i)$

$(x_i \ge \xi_i \ge x_{i-1})$

$$\sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i)\varphi(\xi_i) = \sum_{i=1}^{n} \varphi(\xi_i)(F(x_i) - F(x_{i-1}))$$

$$= \sum_{i=1}^{n-1} F(x_i)(\varphi(\xi_i) - \varphi(\xi_{i+1})) + \varphi(\xi_n) F(b) - \varphi(\xi_1) F(a)$$

[illegible] limit [illegible] $\int f \varphi\,dx$

[illegible] $\varphi(a+0)(F(\xi) - F(a)) = F(b)\varphi(b-0)$

$+ \varphi(b-0)(F(b) - F(\xi)) - F(a)\varphi(a+0)$

$+ F(\xi)(\varphi(a+0) - \varphi(b-0))$

[illegible] $\sum_{i=1}^{n-1} F(\xi)(\varphi(\xi_i) - \varphi(\xi_{i+1}))$ [illegible] ~~monotonic~~

φ monotonic [illegible]

$\sum_{i=1}^{n-1} F(\xi)(\varphi(\xi_i) - \varphi(\xi_{i+1}))$ [illegible]

$F(\xi)(\varphi(\xi_1) - \varphi(\xi_n))$ [illegible]

limit [illegible] $\xi_1 \to a+0$ $\xi_n \to b-0$ [illegible]

(a, b) $\varphi(x)$ monotonous $f(x) = \frac{\sin mx}{x}$ [illegible]

[illegible] theorem [illegible] $\int_a^b \frac{\sin mx}{x}\varphi(x)\,dx = \varphi(a+0)\int_a^{\xi} \frac{\sin mu}{u}\,du$

$+ \varphi(b-0)\int_{\xi}^{b} \frac{\sin mu}{u}\,du$

$$\int_\alpha^\beta \frac{\sin mu}{u}\,du = \int_{\alpha m}^{\beta m} \frac{\sin v}{v}\,dv = \int_0^{\beta m} \frac{\sin v}{v}\,dv - \int_0^{\alpha m} \frac{\sin v}{v}\,dv \qquad v = mu$$

if $\int_0^\infty \frac{\sin v}{v}\,dv = c$

$m \to \infty$ $\int_0^\infty \frac{\sin v}{v}\,dv - \int_0^\infty \frac{\sin v}{v}\,dv = 0$

[illegible] $\int_0^\infty \frac{\sin v}{v}\,dv = \frac{\pi}{2}$

$$J = \int_0^\infty e^{-xy} \frac{\sin x}{x}\,dx \qquad (y > 0)$$

$$= \int_0^\infty \sin x\,dx \int_y^\infty e^{-yx}\,dy = \int_y^\infty dy \int_0^\infty e^{-yx}\sin x\,dx$$

$$\int_0^\infty e^{-yx}\sin x\,dx = \left(-\frac{e^{-yx}}{y}\sin x\right)_0^\infty + \int_0^\infty \frac{e^{-yx}\cos x\,dx}{y}$$

$$= \left(-\frac{e^{-yx}}{y^2}\cos x\right)_0^\infty - \int_0^\infty \frac{e^{-yx}}{y^2}\sin x\,dx$$

$$= \frac{1}{y^2} - \frac{1}{y^2}\int_0^\infty e^{-yx}\sin x\,dx$$

$$(1+y^2)\int_0^\infty e^{-yx}\sin x\,dx = 1$$

$$\therefore \int_0^\infty e^{-yx}\sin x\,dx = \frac{dy}{1+y^2}$$

$$J = \int_y^\infty \frac{dy}{1+y^2} = (\arctan y)_y^\infty = \frac{\pi}{2} - \arctan y$$

$y \to 0$ [illegible] $e^{-xy} = 1$. $\arctan y = 0$

$$\therefore \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

$$\lim_{m\to\infty}\int_\alpha^\beta \frac{\sin mu}{u}\,du = 0 \qquad \alpha>0$$

$$\text{"} \qquad = \frac{\pi}{2} \qquad \alpha = 0$$

$$\int_\alpha^\beta \varphi(u)\frac{\sin mu}{u}\,du = \varphi(\alpha+0)\int_\alpha^\xi \frac{\sin mu}{u}\,du + \varphi(\beta-0)\int_\xi^\beta \frac{\sin mu}{u}\,du$$

$$\lim_{m\to\infty}\int_\alpha^\beta \varphi(u)\frac{\sin mu}{u}\,du = 0 \qquad \alpha>0$$

$$\text{"} \qquad = \frac{\pi}{2}\varphi(+0) \qquad \alpha = 0$$

$$\lim_{m\to\infty}\int_\alpha^\beta \varphi(u)\frac{\sin m(u-x)}{u-x}\,du = \lim_{m\to\infty}\left[\int_x^\beta - \int_x^\alpha\right]$$

$u-x=v$
$$= \lim_{m\to\infty}\left[\int_0^{\beta-x}\varphi(v+x)\frac{\sin mv}{v}\,dv - \int_0^{\alpha-x}\varphi(v+x)\frac{\sin mv}{v}\,dv\right]$$

$\alpha<x<\beta$ v, sign [illegible] 故に
$$= \lim_{m\to\infty}\left[\int_0^{\beta-x}\varphi(v+x)\frac{\sin mv}{v}\,dv + \int_0^{x-\alpha}\varphi(x-v)\frac{\sin mv}{v}\,dv\right]$$

$$= \frac{\pi}{2}\varphi(x+0) + \frac{\pi}{2}\varphi(x-0)$$

[illegible]

$$\frac{\sin m(u-x)}{u-x} = \int_0^m \cos t(u-x)\,dt$$

φ が α, β の間に monotonic [illegible] finite no. の max or min [illegible] (interval [illegible]) or infinitely oscillate [illegible]

故に $\alpha=-\infty$, $\beta=+\infty$ [illegible] x は any no.

$$\lim_{m\to\infty}\int_{-\infty}^{+\infty}\varphi(u)\,du\int_0^m \cos t(u-x)\,dt = \pi\,\frac{\varphi(x+0)+\varphi(x-0)}{2}.$$

φ の order of integration [illegible]

$$\lim_{m\to\infty}\int_0^m dt\int_{-\infty}^{+\infty}\varphi(u)\cos t(u-x)\,dt = \int_0^\infty dt\int_{-\infty}^{+\infty}\varphi(u)\cos t(u-x)\,du$$

$$\frac{1}{\pi}\int_0^\infty dt\int_{-\infty}^{+\infty}\varphi(u)\cos t(u-x)\,du = \frac{\varphi(x+0)+\varphi(x-0)}{2}$$

$\varphi(u)$ continuous at $u=x$ [illegible] $\dfrac{\varphi(x+0)+\varphi(x-0)}{2} = \varphi(x)$

$$\frac{1}{\pi}\int_0^\infty dt\int_{-\infty}^{+\infty}\varphi(u)\cos t(u-x)\,du = \varphi(x) \quad \text{is}$$

Fourier's double integral [illegible]

$\varphi(x)$ one valued, discont. finite, finite extremum [illegible] sign [illegible]

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty}\varphi(u)\cos t(u-x)\,du = \frac{\varphi(x+0)+\varphi(x-0)}{2}$$

$$\int_0^\infty \underbrace{\cos t(u-x)}_{H(x,u,t)}\,dt = K(u,x)$$

[illegible] $\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty}\varphi(u)K(x,u)\,du = \varphi(x)$

$\varphi(x)$; unknown f. [illegible] ~~double~~ integral equation [illegible]
Fourier's double integral [illegible] integ. eq. of the 1st kind
[illegible] int. eq. ; solution [illegible]
[illegible] cond. of heat infinite length [illegible]

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

init. cond. $u = f(x)$ for $t=0$, $0<x<\infty$

$u=0$ for $x=0$

special sol. $e^{-c^2\lambda^2 t}\sin\lambda x$

$$u = \int_0^\infty F(\lambda) e^{-c^2\lambda^2 t}\sin\lambda x \, d\lambda$$ infinite series ... λ parameter integral

init. cond. ... $\lambda = \frac{n\pi}{l}$ ... discont. ... λ cont.

$$\int_0^\infty F(\lambda)\sin\lambda x \, d\lambda = f(x)$$ integ. eq. ... $F(\lambda)$ ...

Fourier's double int. ... $\varphi(u) = f(x)$ ...

$$f(x) = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{+\infty} f(u)\cos\lambda(u-x)\,du$$

$$= \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{+\infty} f(u)\{\cos\lambda u\cos\lambda x + \sin\lambda u \sin\lambda x\}\,du$$

$$\int_{-\infty}^{+\infty} f(u)\cos\lambda u\cos\lambda x\,du$$

$f(x) = -f(-x)$ odd funct. ...

$$= \int_{-\infty}^0 + \int_0^{+\infty} = \int_\infty^0 + \int_0^\infty = 0$$

$$f(x) = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{+\infty} f(u)\sin\lambda u\sin\lambda x\,du$$

$$= \frac{1}{\pi}\int_0^\infty \sin\lambda x\,d\lambda \int_{-\infty}^{+\infty} f(u)\sin\lambda u\,du$$

$$\therefore F(\lambda) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\sin\lambda u\,du$$ ...

$\int_0^\infty \sin\lambda x\,F(\lambda)\,d\lambda = f(x)$ integral eq. ... solution

$\int_a^b K(x,y)\,\underset{\text{unknown}}{\varphi(y)}\,dy = f(x)$ int. eq. of the 1st (kind) ... definite integ. inversion

... int. eq. of the 2nd kind ...

$$u = \frac{2}{\pi}\int_0^\infty d\lambda \int_0^\infty f(u)\sin\lambda u\sin\lambda x\, e^{-c^2\lambda^2 t}\,du$$ sol.

$$= \frac{2}{\pi}\int_0^\infty d\lambda \int_0^\infty f(u)\left(\cos\lambda(u-x) - \cos\lambda(u+x)\right)e^{-c^2\lambda^2 t}\,du$$

$$= \frac{1}{2c\sqrt{\pi t}}\int_0^\infty f(u)\left[e^{-\frac{(x-u)^2}{4c^2t}} - e^{-\frac{(x+u)^2}{4c^2t}}\right]du$$

Ex. Abel's integral equation & its generalization

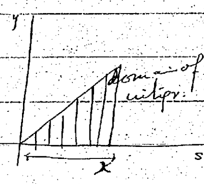
$$f(x) = \int_0^x \frac{\varphi(y)\,dy}{(x-y)^\lambda} \qquad 0<\lambda<1$$

$\varphi(y)$ unknown

Abel's int. eq. ... $f(x)$ known

$$f(s) = \int_0^s \frac{\varphi(y)\,dy}{(s-y)^\lambda}$$

$\frac{ds}{(x-s)^{1-\lambda}}$ ... integrate



$$\int_0^x \frac{f(s)\,ds}{(x-s)^{1-\lambda}} = \int_0^x \frac{ds}{(x-s)^{1-\lambda}}\int_0^s \frac{\varphi(y)\,dy}{(s-y)^\lambda}$$

$$= \int_0^x \varphi(y)\,dy \int_y^x \frac{ds}{(x-s)^{1-\lambda}(s-y)^\lambda}$$

$$\int_y^x \frac{ds}{(x-s)^{1-\lambda}(s-y)^\lambda} \qquad u = \frac{s-y}{x-y}$$

$$= \int_0^1 u^{-\lambda}(1-u)^{\lambda-1}\,du = \frac{\pi}{\sin\lambda\pi}$$ [illegible] γ funct. [illegible]

$$\int_0^x \frac{f(s)\,ds}{(x-s)^{1-\lambda}} = \frac{\pi}{\sin\lambda\pi}\int_0^x \varphi(y)\,dy$$

両辺 [illegible]

$$\varphi(x) = \frac{\sin\lambda\pi}{\pi}\,\frac{d}{dx}\left[\int_0^x \frac{f(s)\,ds}{(x-s)^{1-\lambda}}\right]$$

Abel's integ. eq. [illegible] int. eq. [illegible]

[illegible] generalize [illegible] int. eq. [illegible]

$f(x) = \int_0^x \varphi(y)\,K(xy)\,dy$, Volterra's type $\int_0^x$

$f(x) = \int_a^b \varphi(y)\,K(xy)\,dy$ Fredholm's type [illegible]

[illegible] 1st Kind [illegible]

$\varphi(x) + f(x) = \lambda\int \varphi(y)\,K(xy)\,dy$ 2nd Kind

$f(x) = \int_0^x K(x-y)\,\varphi(y)\,dy$ [illegible]

両辺 $\cos t(x-z)$ [illegible] integ [illegible] Abel's int. eq. [illegible]

$$\int_0^\infty f(x)\cos t(x-z)\,dx = \int_0^\infty \cos t(x-z)\,dx\int_0^x K(x-y)\,\varphi(y)\,dy$$

$$= \int_0^\infty \varphi(y)\,dy\int_y^\infty K(x-y)\cos t(x-z)\,dx$$

Put $u(t) = \int_0^\infty K(\lambda)\cos t\lambda\,d\lambda$

$v(t) = \int_0^\infty K(\lambda)\sin t\lambda\,d\lambda$

$$\int_0^\infty f(x)\cos t(x-z)\,dx = u(t)\int_0^\infty \cos t(y-z)\,\varphi(y)\,dy - v(t)\int_0^\infty \sin t(y-z)\,\varphi(y)\,dy$$

$$\int_0^\infty f(x)\sin t(x-z)\,dx = v(t)\int \cdots + u(t)\int \cdots$$

$$\int_0^\infty \varphi(y)\cos t(y-z)\,dy = \int_0^\infty f(x)\,\frac{u(t)\cos t(x-z) + v(t)\sin t(x-z)}{u^2+v^2}\,dx$$

$$\int_0^\infty dt\int_0^\infty \varphi(y)\cos t(y-z)\,dy = \int_0^\infty dt\int_0^\infty f(x) \text{ ——— } dx$$

Fourier's double integral [illegible] $= \pi\varphi(z)$ $\quad 0 < z < +\infty$

左は known f.

[illegible] $\varphi(z)$ known f. [illegible] solve [illegible]

[illegible] 1895 Levi-Civita [illegible] sol. [illegible]

Calculus of variations

従来ノ max. min. ノ問題ニ於テハ

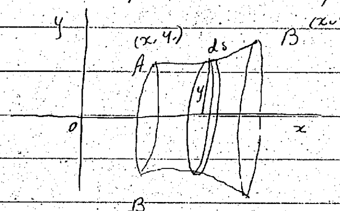
$y = f(x)$ 　 $y = f(x_1, \dots x_n)$

ガ与ヘラレタ時 f ガ extremum トナル様ナ x ガ或ル値ヲ

value ヲ取ル事ヲ求メルノガ普通ノ微分学ノ方法デアル

此等ハ max. min. ノ問題ノ普通

Problem of surface of revol. of minimum area ヲ考ヘル



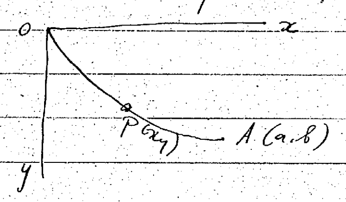
AB ナル任意ノ曲線ヲ Ox axis ノ回リニ回転シテ出来ル area ヲ min. ナラシメル curve ヲ求メヨ

$y = f(x)$ トシ

$$\int_A^B 2\pi y\,ds = 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx = \text{Min}$$

integral ヲ min. トナル様ナ unknown f. ヲ求メル

2 Problem of Brachistochrone



A given. OA ヲ結ブ1 curve ニ沿ッテ material pt. ヲ init. v. 0 デ curve ニ沿ッテ落トシ OA ヲ通過スル時間ヲ min. ニスル curve ヲ求メル

$y = f(x)$

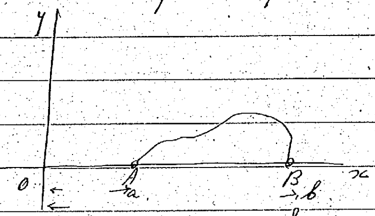
vel. at P 　 $v = \sqrt{2gy} = \frac{ds}{dt}$

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}$$

$$T = \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \text{Min.}$$ ナル y ヲ求ム

此問題ガ classical ナ calculus of v. ノ元デ Johann Bernoulli ノ出シタ問題デアル

~~Problem of~~ Isoperimetric problem



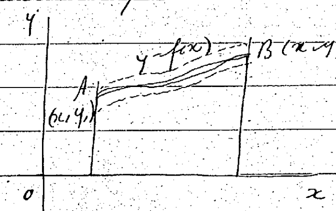
AB ヲ constant length ノ curve トシテ其ノ下ノ area ヲ max. ニセヨ

$y = f(x)$ 　 $\int_a^b y\,dx = \text{Max}$

$\int_a^b \sqrt{1+y'^2}\,dx = \text{const.}$

斯ル問題ヲ取扱フ学問ガ cal. of v. デアル

Necessary cond.s



$J = \int_{x_1}^{x_2} f(x, y, y')\,dx$ ヲ max. min. ニスル

此ガ min. トナル為ノ条件ヲ考ヘル

今 AB ヲ C ナル curve ニ沿ッテ

此 min. トナル条件ヲ analyt. ニ expres...

C: $y = f(x)$ 　 J_C

C ノ両端ヲ固定 domain ...

$y = f(x) + \varepsilon \qquad y = f(x) - \varepsilon$

[illegible] C の curve の vicinity, neighborhood [illegible]

A, B [illegible] domain [illegible] curve [illegible]

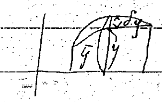
$\bar{C} : y = \bar{f}(x) = \bar{y}$ 2nd. der. [illegible]

funct. one valued cont. differentiable [illegible] $(f, f', f'', \bar{f}, \bar{f}', \bar{f}'')$

$J_C \geq J_{\bar{C}}$ max. $J_C \leq J_{\bar{C}}$ min [illegible]

[illegible]

$|f(x) - \bar{f}(x)| < \varepsilon$ [illegible]



$\bar{y} - y = \bar{f}(x) - f(x) = \delta y$

δy; <u>total variation</u> of y [illegible]

$\bar{y}' - y' = \delta y'$

$J_{\bar{C}} - J_C = \Delta J = \int_{x_1}^{x_2} [f(x, \bar{y}, \bar{y}') - f(x, y, y')]\, dx$

total var. of J

$= \int_{x_1}^{x_2} [f(x, y+\delta y, y'+\delta y') - f(x, y, y')]\, dx$

f Taylor's series [illegible] expansible [illegible]

$f(x, y+\delta y, y'+\delta y') = f(x, y, y') + \left(\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'}\right)$

$+ \frac{1}{2!}\left((\delta y)^2 \frac{\partial^2 f}{\partial y^2} + 2\delta y \delta y' \frac{\partial^2 f}{\partial y \partial y'} + (\delta y')^2 \frac{\partial^2 f}{\partial y'^2}\right)$

$+$

$\delta y = \varepsilon \eta(x)$ [illegible] var. [illegible] $\eta(x)$ one val. cont. $+$

δy [illegible] A, B [illegible] 0 [illegible] η', η'' cont.

$\eta(x) = 0$ at $x = x_1, x_2$

$\Delta J = \varepsilon \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right) dx + \frac{\varepsilon^2}{2} \int_{x_1}^{x_2} \left(\eta^2 \frac{\partial^2 f}{\partial y^2} + 2\eta\eta' \frac{\partial^2 f}{\partial y \partial y'} + \eta'^2 \frac{\partial^2 f}{\partial y'^2}\right) dx$

$+ \cdots$

$= \delta J + \delta^2 J + \cdots$

δJ first variat.

$\delta^2 J$ second 〃 [illegible]

J_C [illegible] extremum [illegible] $\delta J = 0$

[illegible] $\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta'\right) dx = 0$ [illegible] ε [illegible] $\pm$ [illegible] ΔJ [illegible]

[illegible] η [illegible] arbitrary f. [illegible]

[illegible] integrat. by parts

$\int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta\, dx + \left(\frac{\partial f}{\partial y'} \eta\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx = 0$

〃
0

$\int_{x_1}^{x_2} \eta \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx = 0$

$\eta(x)$ [illegible] (x_1, x_2) one valued + cont.

$\eta(x_1) = 0 \quad \eta(x_2) = 0$

η', η'' cont.

[illegible] $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \neq 0$ $x = x_0$ [illegible]

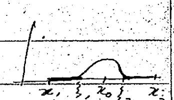
[illegible] x cont. f. [illegible] 0 [illegible]

[illegible] $\neq 0 \quad (\xi_1, \xi_2)$

[illegible] $\eta(x) = 0$ for (x_1, ξ_1)

$= 0$ for (ξ_2, x_2)

$= (x - \xi_1)^4 (\xi_2 - x)^4$ for (ξ_1, ξ_2)



[illegible] η [illegible] discont. one val. [illegible]

η', η'' [illegible] cont. $\eta(x_1) = 0 \quad \eta(x_2) = 0$

故ニ η ヲ用ヒテ $\int_{x_1}^{x_2}\eta\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)dx=0$

$x_1 — x_2$ ヲ $x_1 — \xi_1 — \xi_2 — x_2$ [illegible]

$\int_{\xi_1}^{\xi_2}(x-\xi_1)^4(\xi_2-x_2)^4\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)dx=0$ [illegible]

$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$ ガ (ξ_1, ξ_2) [illegible] $\neq 0$ [illegible]

然ルニ $(x-\xi_1)^4(\xi_2-x)^4$ ハ >0

negative [illegible] sign [illegible]

integral ガ 0 [illegible]

$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$ [illegible]

J ガ max. min. [illegible] dif. eq.

[illegible] Curve ガ [illegible] calculus of var. [illegible]

[illegible] eq. ハ second order [illegible] eq. [illegible]

Euler–Lagrange's equation トイフ

此ハ [illegible] sufficient cond. [illegible]

Ex. [illegible]

1. Shortest distance between two pts.

(x_1, y_1) (x_2, y_2)

$$J=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx = \text{Min.}$$

$$f(x\, y\, y') = \sqrt{1+y'^2}$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)=0$$

$$y' = \text{const} = a$$

$$y = ax+b \quad \text{solut.}$$

Ex. 2. Surface of revolut. of minimum area

$$J = 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx = \text{min.}$$

$$f(x\,y\,y') = y\sqrt{1+y'^2}$$

$$\sqrt{1+y'^2}-\frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)=0$$

$$(1+y'^2) - yy'' = 0$$

$$yy'' = 1+y'^2$$

$$y'=p \qquad \frac{dp}{dy}=\frac{1+p^2}{yp} \qquad \frac{p\,dp}{1+p^2}=\frac{dy}{y}$$

$$\frac{1}{2}\log(1+p^2)=\log y$$

$$y = c\sqrt{1+p^2} \qquad y=\frac{a}{2}\left(e^{\frac{x-c}{a}}+e^{-\frac{x-c}{a}}\right)$$

catenary

一般ニ area ノ min. [illegible] minimal surface トイフ 一般ニ [illegible]

part. dif. eq. of the 2nd order [illegible]

surface of revol. [illegible] minimal [illegible] surface [illegible]

[illegible] catenary [illegible]

[illegible]

Newton's Problem

$$y = \frac{a(1+p^2)^2}{p^3}$$

$$\begin{cases} x = \int \frac{dy}{p} = \frac{y}{p} - \int y \frac{dp}{p^2} \\ \quad = a\left(\frac{3}{4p^4} + \frac{1}{p^2} + \log p\right) + b \\ y = \frac{a(1+p^2)^2}{p^3} \end{cases}$$

(p = parameter)

$$\frac{dx}{dp} = \frac{a(p^2-3)(p^2+1)}{p^5} \qquad \frac{dy}{dp} = \frac{a(p^2-3)(p^2+1)}{p^4}$$

$x = 0$ ... $y = ae^{\frac{x-b}{a}}$ ... cusp ... $p^2 = 3$

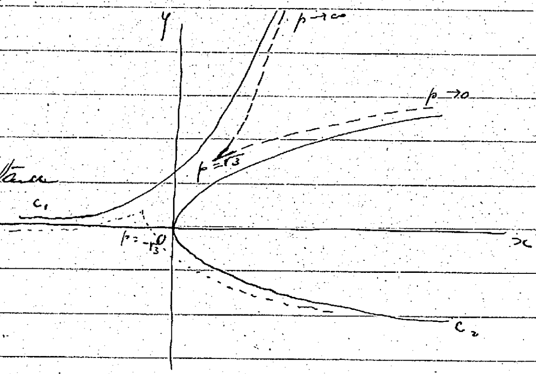
C_1 : $x - b = a \log \frac{y}{a}$ ... $p \to \infty$ ... asymptotic curve

C_2 : $y^4 = \frac{b^4}{2^7} ax^3$ ... $p = 0$ ... asymptotic curve

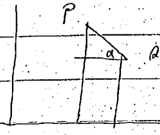
$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dp}\frac{d^2y}{dp^2} - \frac{dy}{dp}\frac{d^2x}{dp^2}}{\left(\frac{dx}{dp}\right)^3} = \frac{1}{\frac{dx}{dp}}$$

$p > \sqrt{3}$, $\frac{d^2y}{dp^2} > 0$

$0 < p < \sqrt{3}$ ... < 0



zigzag line ... resistance



$$J = \text{const} \times \int \sin^3\theta \, y \, ds$$

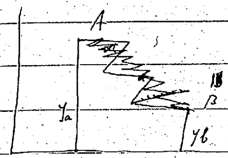
$$J = \int_P^Q \sin^3\alpha \, y \, ds$$

$$dx = ds \cos\alpha$$

$$J = \frac{\sin^3\alpha}{\cos\alpha}\int_P^Q y\,dx = \frac{\sin^3\alpha}{\cos\alpha}\,\frac{y_Q + y_P}{2}\,PQ\cos\alpha$$

$$= \sin^3\alpha \, \frac{y_Q + y_P}{2}\, PQ$$

$$= \sin^2\alpha \, \frac{y_P^2 - y_Q^2}{2}$$



$$J = \frac{1}{2}(y_a^2 - y_b^2)\sin^2\alpha$$

$\alpha \to 0$ ... J ...

Minimum ... curve ... zigzag ... first deriv. discont. ... discontinuous sol. ... Euler–Lagrange ... Calculus of variation ...

Generalisation

$$J = \int_{x_1}^{x_2} f(x, y, y', y'', \dots, y^{(n)})\,dx = \text{Max or Min}$$

$$\begin{cases} y \text{ given at } x = x_1, \; x = x_2 \\ y' \text{ given} \\ y'' \; " \\ - - \\ y^{(n-1)} \end{cases} \qquad \begin{matrix} y & \delta y & \text{variation} \\ y' & \delta y' & \\ - & - & - \\ y^{(n)} & \delta y^{(n)} & \end{matrix}$$

$$\Delta J = \int_{x_1}^{x_2}\left[f(x, y+\delta y, y'+\delta y', \dots, y^{(n)}+\delta y^{(n)}) - f(x, y, y', \dots, y^{(n)})\right]dx$$

$$= \overbrace{\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \cdots + \frac{\partial f}{\partial y^{(n)}}\delta y^{(n)}\right)dx}^{\delta J} + \cdots$$

$\delta y = \varepsilon\eta(x)$ [illegible] $\eta, \eta', \cdots \eta^{(n-1)} = 0$ at $x = x_1, x_2$ —

$$\delta J = \varepsilon\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \cdots + \frac{\partial f}{\partial y^{(n)}}\delta\eta^{(n)}\right)dx = 0$$

$\int_{x_1}^{x_2}\frac{\partial f}{\partial y^{(k)}}\eta^{(k)}dx$ [illegible]

$$= \underbrace{\left(\frac{\partial f}{\partial y^{(k)}}\eta^{(k-1)}\right)_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y^{(k)}}\right)\eta^{(k-1)}dx$$

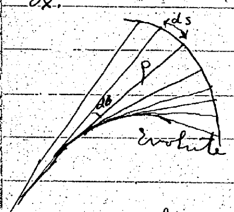
$$= \underbrace{-\left(\frac{d}{dx}(\ )\eta^{(k-2)}\right)_{x_1}^{x_2}}_{0} + \int_{x_1}^{x_2}\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y^{(k)}}\right)\eta^{(k-2)}dx$$

$$= (-1)^k\int_{x_1}^{x_2}\frac{d^k}{dx^k}\left(\frac{\partial f}{\partial y^{(k)}}\right)\eta\, dx$$

$$\delta J = \varepsilon\int_{x_1}^{x_2}\eta\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \cdots + (-1)^n\frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right)\right]dx$$

[illegible] η [illegible] [ ] $= 0$ [illegible] at most $2n$ order dif. eq. [illegible] Euler-Lagrange's equation [illegible]

Ex.



Curve [illegible] evolute [illegible] area [illegible] min [illegible]

$$\int_{\theta_1}^{\theta_2}\frac{1}{2}\rho^2\,d\theta \qquad \rho\,d\theta = ds$$

$$= \frac{1}{2}\int_{\theta_1}^{\theta_2}\rho\,ds$$

$y = f(x)$ $\quad \rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}$ $\quad ds = \sqrt{1+y'^2}\,dx$

$$J = \frac{1}{2}\int_{x_1}^{x_2}\frac{(1+y'^2)^2}{y''}dx = Min$$

$$\underbrace{\frac{\partial f}{\partial y}}_{0} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = a$$

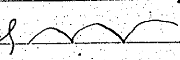
$$\frac{d}{dx}\left(f - y''\frac{\partial f}{\partial y''}\right) = ay''$$

$f - y''\frac{\partial f}{\partial y''} = ay' + b$ $\quad x=0,\ y=0$ [illegible]

$$\frac{(1+y'^2)^2}{y''} + y''\frac{(1+y'^2)^2}{y''^2} = \frac{2(1+y'^2)^2}{y''} = ay' + b$$

$$\frac{(1+y'^2)^2}{y''} = \rho\frac{ds}{dx} = \frac{ay'+b}{2}$$

$y' = u$ $\quad \frac{(1+u^2)^2}{u'} = \frac{au+b}{2}$

$u' = \frac{2(1+u^2)^2}{au+b}$ $\quad$ cycloid 

Curvature [illegible] y'' [illegible] $\int f(x, y, y', y'')dx$ [illegible]

$\int\frac{ds}{\rho^2}$ [illegible] max. min. [illegible] elastic curve [illegible]

Another Generalisation

Dependent variable [illegible]

$$J = \int_{x_1}^{x_2} f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right)dx = \text{max or min}$$

y, z … x … funct … x_1, x_2 … $y\ z$ … [illegible]
… space … two points … space curve
… curve … integral … max
or min … curve … [illegible]
y, z … indep. … variate … [illegible]
… y … vary … necess cond

$$\left(\begin{array}{l} \dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0 \qquad y' = \dfrac{dy}{dx} \\ y \text{ … var. …} \\ \dfrac{\partial f}{\partial z} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial z'}\right) = 0 \end{array}\right.$$

… necessary cond … y, z …
x_1, x_2 … sol …
shortest distance in space

$$J = \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx$$

$$\frac{y'}{\sqrt{\ }} = a \qquad \frac{z'}{\sqrt{\ }} = b$$

… y', z' const. … in …

$$y = \alpha x + \beta$$
$$z = \gamma x + \delta \qquad \text{straight line}$$

dependent v. … [illegible]

$$J = \int_{x_1}^{x_2} (x, y_1, y_2 \dots y_n, y_1', y_2', \dots y_n')\, dx = \text{Min}$$

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_k'}\right) = 0 \qquad k = 1, 2, \dots n$$

… Hamilton's principle …

$$J = \int_{t_1}^{t_2} (T - U)\, dt = \text{Min} \qquad T \text{ kinetic energy} = \varphi(q_1, q_2, \dots q_n)$$

$\delta J = 0$ is necessary $\qquad U$ potential f.

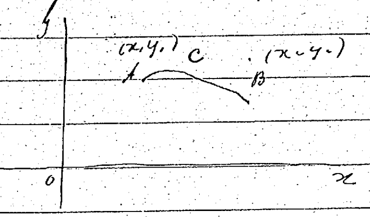
$$\frac{\partial (T-U)}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot q_k}\right) = 0 \qquad U = \psi(q_1, q_2, \dots q_n)$$

… equat. of mot. …

$$-\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_k}\right) = 0 \qquad T = \sum \tfrac{1}{2} m_i (\dot x_i^2 + \dot y_i^2 + \dot z_i^2)$$

dynamic principle … minimum … [illegible]
… math. min … physical fact … [illegible]
… dep. v. … higher order derivative … [illegible]
… cases …

Isoperimetric problem



$$J = \int_{x_1}^{x_2} f(x, y, y')\, dx = \text{Min}$$

$$K = \int_{x_1}^{x_2} g(x, y, y')\, dx = \text{Const.}$$

Isoperimetric … $\int g$ … curve

length ... area ... sum

C $y = f(x)$ ... sol. ...

... $\bar{C}$ ... $\bar{y} = \bar{f}(x)$

$J_{\bar{C}} - J_C = \Delta J \geq 0$

$K_{\bar{C}} - K_C = \Delta K = 0$

$\bar{y} - y = \delta y \quad \bar{y}' - y' = \delta y'$ ... variation ... $\bar{C}$ ...

$\bar{y} = y + \delta y$

$\bar{y}' = y' + \delta y'$

$\Delta J = \delta J + \delta^2 J + \cdots$

$\Delta K = \delta K + \delta^2 K + \cdots$

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx$$

$$\delta K = \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial y'} \delta y' \right) dx$$

$\delta y = \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x)$

$\eta_1(x), \eta_2(x) = 0$ at $x = x_1, x_2$

$$\delta J = \varepsilon_1 \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta_1 + \frac{\partial f}{\partial y'} \eta_1' \right) dx + \varepsilon_2 \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta_2 + \frac{\partial f}{\partial y'} \eta_2' \right) dx$$

$$\delta K = \varepsilon_1 \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \eta_1 + \frac{\partial g}{\partial y'} \eta_1' \right) dx + \varepsilon_2 \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \eta_2 + \frac{\partial f}{\partial y'} \eta_2' \right) dx$$

$\delta^2 J = (\varepsilon_1, \varepsilon_2)_2$

$\delta^2 K = (\varepsilon_1, \varepsilon_2)_2$ ... term ... δJ ...

δJ ... ΔJ ... sign ...

$\varepsilon_1, \varepsilon_2$ ... sign ... ΔJ ... sign ...

sign ... $\delta J = 0 \quad \delta K = 0$

$\Delta J = f(\varepsilon_1, \varepsilon_2) \geq 0$

$\Delta K = G(\varepsilon_1, \varepsilon_2) = 0$

$$\varepsilon_1 \int \eta_1 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx + \varepsilon_2 \int \eta_2 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx = 0$$

$$\varepsilon_1 \int \eta_1 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right) dx + \varepsilon_2 \int \eta_2 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right) dx = 0$$

$$\cancel{\varepsilon_1 \int \eta_1 \left\{ \frac{\partial (f + \lambda g)}{\partial y} - \frac{d}{dx}\left(\frac{\partial (f+\lambda g)}{\partial y'}\right) \right\} dx +}$$

$\varepsilon_1, \varepsilon_2$ ... eliminate ...

$$\frac{\int \eta_1 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx}{\int \eta_1 \left(\frac{\partial g}{} \quad \frac{\partial g}{} \right) dx} = \frac{\int \eta_2 \left(\quad \right) dx}{\int \eta_2 \left(\ g \quad g \ \right) dx} = -\lambda$$

$$\cancel{\int \eta_1 \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx - d}$$

$$\int \eta_1 \left\{ \frac{\partial (f + \lambda g)}{\partial y} - \frac{d}{dx}\left(\frac{\partial (f + \lambda g)}{\partial y'} \right) \right\} dx = 0$$

$$\int \eta_2 \left\{ \qquad \right\} dx = 0$$

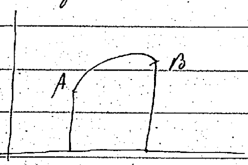
$$\frac{\partial (f + \lambda g)}{\partial y} - \frac{d}{dx}\left(\frac{\partial (f + \lambda g)}{\partial y'} \right) = 0 \qquad \lambda \text{ const.}$$

... dif eq. ... solution ... λ ... $K = \text{const}$

... eq. ... Euler-Lagrange's eq ...

Ex. $J=\int_{x_1}^{x_2} y\,dx$ $\qquad K=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$

definite curve length → $\overset{\text{max}}{\text{min}}$ area [illegible]



$$f+\lambda g = y+\lambda\sqrt{1+y'^2}$$

$$\frac{\partial(f+\lambda g)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(f+\lambda g)}{\partial y'}\right)=0$$

$$1-\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)=0$$

$$\frac{1}{\lambda}=\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)$$

$$c+\frac{x}{\lambda}=\frac{y'}{\sqrt{1+y'^2}}$$

$$y'=\frac{x+c\lambda}{\sqrt{\lambda^2-(x+c\lambda)^2}}$$

$$y+\text{Const}=-\sqrt{\lambda^2-(x+c\lambda)^2}$$

$$(y-c')^2=\lambda^2-(x+c\lambda)^2$$

$(x+c\lambda)^2+(y-c')^2=\lambda^2$ circle, λ radius

i.e. circle [illegible] A, B [illegible] pass [illegible] arc length given.

max, min [illegible] circular arc [illegible]

~~Ex.~~ $\cancel{J=\int_{x_1}^{x_2} f(x\,y\,y')\,dx=\text{Min}}$

$\cancel{g(x\,y\,y')=0}$ auxiliary cond. integral [illegible]

$$J=\int_{x_1}^{x_2} f(x\,y\,z\,y'\,z')\,dx=\text{Min} \qquad g(xyz)=0$$

[illegible]

i.e. space two pts [illegible] space curve [illegible] curve [illegible] integral [illegible] min [illegible] curve [illegible] $g(xyz)=0$ [illegible] surface [illegible]

$$z=\varphi(xy) \qquad F(x\,y\,y')$$

$$J=\int_{x_1}^{x_2} f\left(x,y,\varphi(xy),\ y',\ \frac{\partial f}{\partial x}+\frac{\partial\varphi}{\partial y}y'\right)dx=\text{Min}$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

$$\frac{\partial F}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial\varphi}{\partial y}+\frac{\partial f}{\partial z'}\left(\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial y^2}y'\right)$$

$$\frac{\partial F}{\partial y'}=\frac{\partial f}{\partial y'}+\frac{\partial f}{\partial z'}\frac{\partial\varphi}{\partial y}$$

$$=\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial\varphi}{\partial y}\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)+\frac{\partial f}{\partial z'}\left(\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial y^2}y'\right)$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial\varphi}{\partial y}\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)=0$$

[illegible] $\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}\frac{\partial\varphi}{\partial y}=0$

[illegible] $\frac{\partial\varphi}{\partial y}$ [illegible]

$$\frac{\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}{\frac{\partial g}{\partial y}}=\frac{\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)}{\frac{\partial g}{\partial z}}=-\lambda$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\lambda\frac{\partial g}{\partial y}=0 \qquad \frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)+\lambda\frac{\partial g}{\partial z}=0$$

$\lambda = \lambda(x)$

$$\frac{\partial(f+\lambda g)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(f+\lambda g)}{\partial y'}\right) = 0$$

$$\frac{\partial(f+\lambda g)}{\partial z} - \frac{d}{dx}\left(\frac{\partial(f+\lambda g)}{\partial z'}\right) = 0$$

} necessary condit.

Ex. $f(x\,y\,z)=0$ surface [illegible] two given pts [illegible] shortest distance [illegible]

$$J = \int_{x_1}^{x_2} \sqrt{1+y'^2+z'^2}\, dx = \text{Min.}$$

$$\frac{\partial(\sqrt{1+y'^2+z'^2}+\lambda f)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(\sqrt{\quad}+\lambda f)}{\partial y'}\right) = 0$$

∂z $\qquad$ $\partial z'$

$$\lambda\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\lambda\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{z'}{\sqrt{\quad}}\right) = 0$$

[illegible]

$x = x(s)$

$y = y(s)$

$z = z(s)$ [illegible]

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} \qquad z' = \frac{dz}{dx} = \frac{\frac{dz}{ds}}{\frac{dx}{ds}}$$

$$1+y'^2+z'^2 = \frac{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2}{\left(\frac{dx}{ds}\right)^2} = \frac{1}{\left(\frac{dx}{ds}\right)^2}$$

$$\lambda\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{dy}{ds}\right) = 0 = \lambda\frac{\partial f}{\partial y} - \frac{\frac{d}{ds}\left(\frac{dy}{ds}\right)}{\frac{dx}{ds}} = 0$$

$$\lambda\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{dz}{ds}\right) = 0 = \lambda\frac{\partial f}{\partial z} - \frac{\frac{d}{ds}\left(\frac{dz}{ds}\right)}{\frac{dx}{ds}} = 0$$

λ を elimination

$$\frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} = \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} \qquad \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

curve [illegible] principal normal $\qquad$ $f=0$ [illegible] surface normal [illegible]

[illegible] direction cosine [illegible] proport. $\qquad$ direction cosine [illegible] proport.

[illegible] curve [illegible] principal normal [illegible] surface normal [illegible] geodesic line [illegible]

Ex. Hamilton's principle

$$\int_{t_1}^{t_2} (T-U)\,dt = \text{Min.}$$

$$T = \sum \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2), \quad U = U(x, y, z, \ldots)$$

particle [illegible] surface [illegible] $f(x\,y\,z)=0$

[illegible] particle [illegible] motion [illegible]

$$\frac{\partial((T-U)+\lambda f)}{\partial x} - \frac{d}{dt}\left(\frac{\partial(\quad)}{\partial \dot{x}}\right) = 0$$

$$-\frac{\partial U}{\partial x} + \lambda\frac{\partial f}{\partial x} - \frac{d}{dt}m\dot{x} = 0 \qquad \text{equation of motion}$$

$$m\ddot{x} = -\frac{\partial U}{\partial x} + \lambda\frac{\partial f}{\partial x}$$

$m\ddot{y} =$

$m\ddot{z} = \ldots$

ここで sufficient cond'n を導くため

The second Variation

$$J=\int_{x_1}^{x_2} f(x,y,y')\,dx = \text{Min}$$

C: $y=f(x)$

$\bar C$: $\bar y = \bar f(x) = y+\delta y$

$$J_{\bar C}-J_C = \Delta J = \delta J + \delta^2 J + \cdots$$

$$\delta^2 J = \frac{1}{2}\int_{x_1}^{x_2}\left\{\frac{\partial^2 f}{\partial y^2}(\delta y)^2 + \frac{\partial^2 f}{\partial y\partial y'}\delta y\,\delta y' + \frac{\partial^2 f}{\partial y'^2}(\delta y')^2\right\}dx$$

$\delta y = \varepsilon\eta(x)$

$$\delta^2 J = \frac{\varepsilon^2}{2}\int_{x_1}^{x_2}\Big(\underbrace{\frac{\partial^2 f}{\partial y^2}}_{P}\eta^2 + 2\underbrace{\frac{\partial^2 f}{\partial y\partial y'}}_{Q}\eta\eta' + \underbrace{\frac{\partial^2 f}{\partial y'^2}}_{R}\eta'^2\Big)dx$$

$$= \frac{\varepsilon^2}{2}\int_{x_1}^{x_2}\{P\eta^2+2Q\eta\eta'+R\eta'^2\}dx$$

ここで negligible とする

$$\Delta J = \frac{\varepsilon^2}{2}(\quad) + (\varepsilon)_3$$

$\Delta J \geqq 0$ となるには $\delta^2 J\geqq 0$ min necessary

この部分を書き直す ... Jacobi による.

$w(x)$ $w'(x)$ (x_1, x_2) にて cont. f.

$$\int_{x_1}^{x_2}\frac{d}{dx}(w\eta^2)\,dx = (w\eta^2)_{x_1}^{x_2}=0$$

$$=\int_{x_1}^{x_2}(2\eta\eta' w+\eta^2 w')\,dx = 0$$

$$\delta^2 J = \varepsilon^2\int_{x_1}^{x_2}\left((P+w')\eta^2+2(Q+w)\eta\eta'+R\eta'^2\right)dx$$

η' について完全平方 ... $R\left((\eta'+\frac{Q+w}{R}\eta)^2 + (\frac{P+w'}{R} - \frac{(Q+w)^2}{R^2})\eta^2\right)$

$R(P+w')-(Q+w)^2=0$ とすると

$$=\varepsilon^2\int_{x_1}^{x_2} R\left(\eta'+\frac{Q+w}{R}\eta\right)^2dx$$

Riccati's dif. eq., special case

w を $R(P+w')-(Q+w)^2=0$ ...

R positive ならば $\delta^2 J \geqq 0$ となる

$$R=\frac{\partial^2 f}{\partial y'^2}>0 \quad (x_1, x_2)$$

$w(x)$ ... x_1, x_2 にて w, w' は cont. な sol. をもつ ... u の eq. に transform

$w = -Q - R\frac{u'}{u}$ $\qquad u(x)$ とおく

$Q+w = -R\frac{u'}{u}$

$$w' = -Q' - \frac{\frac{d}{dx}(Ru')}{u} + \frac{u'^2 R}{u^2}$$

この eq. は $u(P-Q') - \frac{d}{dx}(Ru')=0$ となる

$$\delta^2 J = \varepsilon^2\int_{x_1}^{x_2} R\left(\frac{u\eta'-u'\eta}{u}\right)^2dx$$

w の eq. は 1st order なるも non-linear ... u の eq. は 2nd order なるも linear homog. ... sol. ... x_1, x_2 にて ...

$u\neq 0$ ...

尚 Euler-Lag. eq. によると

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

ノ general sol. ニ二ツノ arbitrary const. ヲ含ム

$y = \varphi(x, \alpha, \beta)$ extremals トナル

J ヲ minimum ニスル sol. ハ sol. ナルヲ要ス

$y = \varphi(x, \alpha_0, \beta_0)$ トシ

$$\frac{\partial f(x, \varphi, \varphi_x)}{\partial y} - \frac{d}{dx}\left(\frac{\partial f(x, \varphi, \varphi_x)}{\partial y'}\right) = 0$$

之ヲ α, β ニツキ differentiate シ

$$\frac{\partial^2 f}{\partial y^2}\frac{\partial \varphi}{\partial \alpha} + \frac{\partial f}{\partial y \partial y'}\frac{\partial \varphi_x}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\frac{\partial \varphi}{\partial \alpha} + \frac{\partial f}{\partial y'^2}\frac{\partial \varphi_x}{\partial \alpha}\right) = 0$$

$$\frac{\partial \varphi}{\partial \alpha}\left(\frac{\partial f}{\partial y^2} - \frac{d}{dx}\left(\frac{\partial f}{\partial y \partial y'}\right)\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'^2}\frac{\partial \varphi_x}{\partial \alpha}\right) = 0$$

$$\frac{\partial \varphi}{\partial \beta}\left(\quad '' \quad\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'^2}\frac{\partial \varphi_x}{\partial \beta}\right) = 0$$

此ハ即 $\frac{\partial \varphi}{\partial \alpha}(P - Q') - \frac{d}{dx}\left(R\frac{\partial \varphi_x}{\partial \alpha}\right) = 0$

$$\frac{\partial \varphi}{\partial \beta}(P - Q') - \frac{d}{dx}\left(R\frac{\partial \varphi_x}{\partial \beta}\right) = 0$$

故ニ $u = \frac{\partial \varphi}{\partial \alpha}, \frac{\partial \varphi}{\partial \beta}$ ト [illegible]

$$u = \left(\frac{\partial \varphi}{\partial \alpha}\right)_0 = r_1(x)$$

$$\left(\frac{\partial \varphi}{\partial \beta}\right)_0 = r_2(x)$$

general sol. $c_1 r_1(x) + c_2 r_2(x) = u$

此 sol. ノ中 c_1, c_2 ヲ [illegible] $\overbrace{u \text{ ヲ } 0 \text{ トスル}}^{x_1, x_2 \text{ ニテ}}$ sol. [illegible]

$$v = \Delta(x, x_1) = \begin{vmatrix} r_1(x) & r_2(x) \\ r_1(x_1) & r_2(x_1) \end{vmatrix} = r_2(x_1)r_1(x) - r_1(x_1)r_2(x)$$

$r_1(x_1), r_2(x_1)$ const. 故ニ $\Delta(x, x_1)$ ハ u ノ part. sol. ナリ

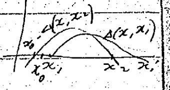
$x = x_1 \quad \Delta = 0$

x ガ x_1 ヨリ増シテ $\Delta(x, x_1)$ ガ 0 トナル最初ノ値 $x = x_1'$

~~$x_1' > x_1$~~ $x_1 < x < x_1' \quad \Delta(x, x_1) \neq 0$

x_1' ガ x_2 ヨリ大ナルカ [illegible]

$x_1' > x_2$ ナラバ $\Delta(x, x_1) \neq 0$ for $x_1 < x \leq x_2$



x_1 ニテ 0 トナル [illegible]

$v(P - Q') - \frac{d}{dx}(Rv') = 0 \qquad u$

$u(P - Q') - \frac{d}{dx}(Ru') = 0 \qquad v$

$$v\frac{d}{dx}(Ru') - u\frac{d}{dx}(Rv') = 0$$

$$\frac{d}{dx}\left(R(u'v - uv')\right) = 0$$

$$R(u'v - uv') = \text{const.}$$

$\Delta(x, x_2) = -\Delta(x_2, x_1) \neq 0$ 故ニ [illegible]

$\Delta(x, x_2) \neq 0$ for $x = x_1$

$\neq 0$ for $x > x_1 - \delta$

$\Delta(x, x_0) = 0 \qquad$ for $x = x_0$

u, v ハ u ノ [illegible] sol. トシ $u'v - uv' = \text{Const.} = K$

$u = \Delta(x, x_2)$

$v = \Delta(x, x_0)$ トシ

$u = 0$ at $x = x_0', x = x_2$, | $u' > 0$ at $x = x_0'$, $u' < 0$ at $x = x_2$

$K = -u'v \qquad$ at $x = x_0', x = x_2$

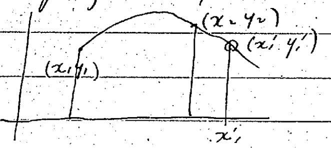
[illegible] u' ハ x_0', x_2 ニテ opposite sign [illegible]
v モ opposite sign [illegible] 故ニ v ハ
x_0', x_2' [illegible] 0 トナル [illegible]
(Sturm's theorem [illegible] second order linear
dif. eq. [illegible]) [illegible] 0 ト [illegible]
[illegible] 0 トナル [illegible] u [illegible]
0 トナル [illegible] (u, v ハ α [illegible])
$v =$
[illegible] $\Delta(x, x_0) = 0$ at $x = x_0$
[illegible] x_0 [illegible] 0 ト [illegible]
[illegible] function [illegible]

(1) $x_1' > x_2$ [illegible] $\delta^2 J = \varepsilon^2 \int_{x_1}^{x_2} R(\ )^2 dx$

(2) [illegible] $R = \frac{\partial^2 F}{\partial y'^2} > 0$ ナラバ $\delta^2 J > 0$

此 [illegible] minimum [illegible] sufficient cond. ナリ
[illegible] Jacobi [illegible] $x_1' > x_2$ ハ Jacobi's condit.
$R = \frac{\partial^2 F}{\partial y'^2} > 0$ ハ Legendre's condit. ナリ

x_1' [illegible] point $x = x_1'$, $y = y_1'$ ハ (x_1, y_1) ノ
conjugate point ナリ



conjugate point [illegible] geometrical [illegible]


TRADE MARK
M H & Co.