NOTE-BOOK.

Synthetic Geometry

II

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Theorem 2. V_1, V_2 … commut … invol … $V_1 V_2$ … invol … invol … prod … invol … commut …

Proof $V_1 V_2 = V_2 V_1$

$(V_1 V_2)^2 = V_1 V_2 \cdot V_2 V_1 = V_1 \cdot V_2^2 \cdot V_1$

$= V_1^2 = 1$ invol …

逆 $V_1 V_2 = V_3$ invol … $V_1^2 = V_2^2 = V_3^2 = 1$.

$V_1 V_2 V_3 = (V_1 V_2)(V_3) = V_3^2 = 1$

$V_2 V_1 = V_2 V_1 \cdot V_1 V_2 V_3 = V_2 V_1^2 V_2 V_3$

$= V_2 \cdot V_2 V_3 = V_2^2 \cdot V_3 = V_3$

$= V_1 V_2$ commutative …

Theorem 3. invol V … conj. p. … $A A'$ … A, A' … doublept … invol V' … V … commut …

Proof $V V'(AA') = V(AA') = (A'A)$ conj …

$(VV')^2(AA') = (VV')(A'A) = (AA')$

$\therefore (VV')^2 = 1$.

… VV' … invol … (2) … commutative …

Literature.

plane, space … invol … cyclic …
… Encyk. d. m. W. III, 1, pp. 434 – 437

Part II.

[illegible] metrically [illegible] proj. geom. [illegible] metrically [illegible] coord. [illegible] figure [illegible] analytic [illegible] homogeneous coord. [illegible] analytic geom. [illegible] systematic [illegible] Heffter-Koehler, Lehrbuch d. analyt. Geom. [illegible] Salmon Conic Sections, Higher Curves, Analyt. G. of 3 dimensions [illegible] proj. geom. [illegible] analyt. algebraic curves algebraic surface [illegible] theorems [illegible] proj. [illegible] differential proj. geom. [illegible] infinitesimal property [illegible] proj. [illegible] proj. method [illegible] proj. geom. [illegible] metrical [illegible] standpt. [illegible] proj. geom. [illegible]

Chapter XIII. Projective Transformations of one dimensional forms

55. Anharmonic ratio & projectivity

segment [illegible] length [illegible] algebraic analysis [illegible] function theory of real variable [illegible]

Def. [illegible] ABCD [illegible]

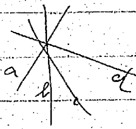
A B C D $\qquad \dfrac{AD}{BC} : \dfrac{AD}{DC}$ [illegible]

anharmonic ratio (Chasles) double ratio (Möbius)
Cross ratio (Clifford) [illegible] $R_x^{\text{cross rat}}(ABCD)$ [illegible]

[illegible] angle [illegible]

Def. flat pencil a b c d [illegible]

$$\frac{\sin(ab)}{\sin(bc)} : \frac{\sin(ad)}{\sin(dc)}$$



[illegible] pencil [illegible] anharm. r. [illegible]

[illegible] axial pencil $\alpha\beta\gamma\delta$ [illegible]

$$\frac{\sin(\alpha\beta)}{\sin(\beta\gamma)} : \frac{\sin(\alpha\delta)}{\sin(\delta\gamma)}$$ [illegible] pencil [illegible] anh. r [illegible]

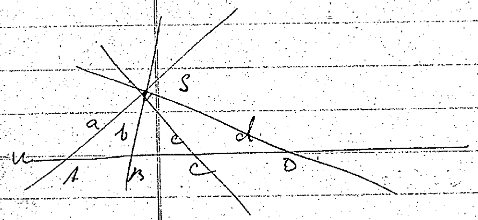
Theorem 1. [illegible] range [illegible] pencil [illegible] corr. el. [illegible] anharm. r. [illegible] (Pappus)

Proof $\triangle ASB = \frac{1}{2} SA \cdot SB \sin(ab)$

$\triangle BSC = \frac{1}{2} SB \cdot SC \sin(bc)$

— — —

. . .



$$\frac{\triangle ASB}{\triangle BSC} = \frac{AB}{BC}$$

———

$$\frac{AB}{BC} : \frac{AD}{DC} = \frac{\sin(ab)}{\sin(bc)} : \frac{\sin(ad)}{\sin(dc)}$$

anharmonic ratio [illegible] proj. + sect. [illegible] invariant [illegible]

Theorem 2. proj. forms of 1st rank [illegible] corr. el. [illegible] anh. r. [illegible] corr. el. [illegible] anh. r. [illegible]

[illegible]

proj. range [illegible] construct [illegible]

ABC D

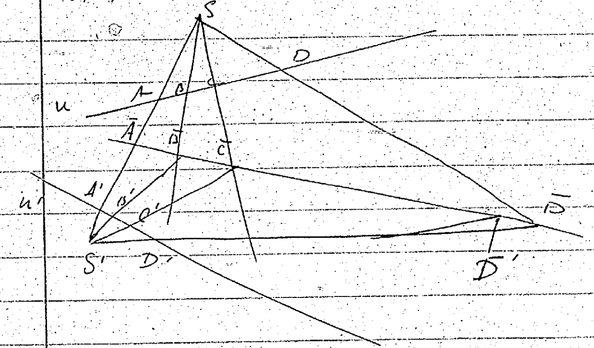
A'B'C' D'

[illegible]

$R_x(ABCD) = R_x(A'B'C'D)$

[illegible] $\bar{D}$, $\bar{D}'$ [illegible]

[illegible]



$R_x(ABCD) = R_x(\bar{A}\bar{B}\bar{C}\bar{D})$

$R_x(A'B'C'D') = R_x(\bar{A}'\bar{B}'\bar{C}\bar{D}')$

$$\therefore \frac{\bar{A}\bar{D}}{\bar{D}\bar{C}} = \frac{\bar{A}\bar{D}'}{\bar{B}\bar{C}}$$

$\bar{A}$, $\bar{C}$ [illegible] $\bar{D}$, $\bar{D}'$ [illegible]

Remark. ... by ... Chasles, Steiner ... projectivity ... defined by v. Staudt, Cremona ...

56. Linear transformat.

I. proj. r. u, u' ... fixed pts O, O' ... A, B, C, X ... A', B', C', X' ...

$OA = a,\ OB = b, \ldots$

O A B C X

$$\frac{AB}{BC} = \frac{AX}{XC} = \frac{b-a}{c-b} : \frac{x-a}{c-x}$$

O' A' B' C' X'

$$\frac{A'B'}{B'C'} = \frac{A'X'}{X'C'} = \frac{b'-a'}{c'-b'} : \frac{x'-a'}{c'-x'}$$

$$\therefore \frac{b'-a'}{c'-b'} : \frac{x'-a'}{c'-x'} = \frac{b-a}{c-b} : \frac{x-a}{c-x}$$

$$x' = \frac{\lambda x + \mu}{\nu x + \rho}$$

λ, μ, ν, ρ constant ... x, x' ...

$\lambda\rho - \nu\mu \neq 0$ ...

$$x = \frac{\lambda' x' + \mu'}{\nu' x' + \rho'}$$

Conv. $x' = \dfrac{\lambda x + \mu}{\nu x + \rho}$, $\lambda\rho - \mu\nu \neq 0$ ...

$$a' = \frac{\lambda a + \mu}{\nu a + \rho},\quad b' = \frac{\lambda b + \mu}{\nu b + \rho},\quad c' = \frac{\lambda c + \mu}{\nu c + \rho}$$

...

$R(A'B'C'X') = R(ABCX)$

...

Theorem. Linear transf. $x' = \dfrac{\lambda x + \mu}{\nu x + \rho}$ ($\lambda\rho - \nu\mu \neq 0$) ... pt range ... projectivity ... necessary & suff. condition ... proj. ... fundamental theorem ... eq.

$\nu x x' - \lambda x + \rho x' - \mu = 0 \quad \lambda\rho - \nu\mu \neq 0$ ...

... bilinear form ...

... λ, μ, ν, ρ ... continuous ... ratio ... transfor ... λ, μ, ν, ρ ... parameters ...

... ∞^3 ... transf ... linear transf. ... linear transf. ... inverse transf. ... exist ... identical transf exist ($\mu = 0, \nu = 0, \frac{\lambda}{\rho} = 1$ ... $x' = x$) ... linear transf ... group ... pt range proj. transf. ... totality ... continuous ~~group~~ ... three-termed groups ... (Lie)

proj. ... double pts ...

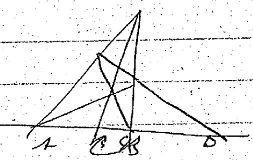
$x' = x \quad \nu x^2 - (\lambda - \rho)x - \mu = 0$

real or imag. pts ...

II. Particular case.

anharm. ... -1 ...

$\frac{AB}{AC} = -\frac{AD}{DC}$



harmonic [illegible]

D [illegible] pt at co [illegible] C [illegible] AD [illegible] middle pt [illegible]

affin geom [illegible] pt at co [illegible] conj. pt [illegible] middle pt [illegible]

Involution [illegible] bilinear eq. [illegible] X, X' [illegible]

symmetrical [illegible]

$u \quad X \qquad \bar{X}'$

$u' \quad \bar{X} \qquad X'$

eq. $-\lambda = \rho$ [illegible]

$\nu x x' + \rho(x + x') - \mu = 0$

$\rho^2 + \mu\nu \gtreqless 0$ hyp. inv. / par. inv. / ellip. inv.

quadratic eq. [illegible]

[illegible] metrically [illegible] root [illegible] imaginary [illegible]

Staudt [illegible] (imaginary) [illegible] 1D [illegible]

Klein [illegible] Elementarmath. II [illegible] 271 — 293. [illegible]

Remark proj. g. [illegible] Fresnin [illegible]

[illegible] Chasles [illegible] Rausch-[illegible]

[illegible] (St. [illegible] Hausch-Trentlin) [illegible] proj. [illegible] synth. geom [illegible]

中[illegible]

67. Notion of the theory of invariants (Binary forms)

linear transf. [illegible]

[illegible] Arith. [illegible] theory of nos. [illegible] quadratic form [illegible] theory [illegible]

analysis [illegible] dif. eq. [illegible] singularity [illegible] automorphic fct. [illegible]

[illegible] Algebra [illegible] Geom.

[illegible] algebra [illegible] linear transf. [illegible] theory [illegible]

invariant theory [illegible] proj. geom. [illegible] standpt. [illegible]

[illegible] pt range [illegible]

$x = \frac{x_1}{x_2}$ [illegible]

(x_1, x_2) [illegible] pt x [illegible] homogeneous coord. [illegible]

[illegible] hom. coord. [illegible] x_1, x_2 [illegible] finite value [illegible]

[illegible] 0 [illegible] $x_2 = 0$ [illegible] pt at co [illegible]

[illegible] linear transf. [illegible]

$$x' = \frac{\lambda x + \mu}{\nu x + \rho}$$

$$\begin{cases} r\rho x_1' = \lambda x_1 + \mu x_2 \\ r\rho x_2' = \nu x_1 + \rho x_2 \end{cases}$$

[illegible] r [illegible] proportional factor

[illegible]

$$\begin{cases} x_1' = \alpha x_1 + \beta x_2 \\ x_2' = \gamma x_1 + \delta x_2 \end{cases}$$

$\Delta = \alpha\delta - \beta\gamma \neq 0$

Δ [illegible] determinant [illegible] substitution [illegible]

$x_1 x_2$ [illegible] homog. linear eq. [illegible]

$a_0 x_1 + a_1 x_2 = 0$

$\frac{x_1}{x_2} = -\frac{a_1}{a_0}$ [illegible] one pt [illegible]

quadr. binary eq. [illegible] two pts on line [illegible]

[illegible] n th deg. homog. algebraic eq. [illegible]

$f = a_0 x_1^n + \frac{n}{1} a_1 x_1^{n-1} x_2 + \frac{n(n-1)}{2!} a_2 x_1^{n-2} x_2^2 + \dots + a_n x_2^n = 0$

[illegible] n pts system on a line [illegible]

[illegible] n th degree [illegible] **binary form** [illegible]

Def. I [illegible] binary form [illegible] coeff. [illegible] rat. integral funct. [illegible] linear transf. [illegible] constant factor [illegible] binary form [illegible] **invariant** [illegible]

$I(a_0, a_1, a_2, \dots, a_n)$ [illegible] rat. int. f. [illegible] linear tr. [illegible] coeff $(a_0' a_1' a_2' \dots a_n')$

[illegible] I [illegible]

$I(a_0' a_1' a_2' \dots a_n') = k\, I(a_0\, a_1 \dots a_n)$

(k [illegible] Δ^λ [illegible])

[illegible] invariant [illegible] λ [illegible] **index of invariant** [illegible]

Binary form [illegible] invariant [illegible] pt system [illegible] projective property [illegible]

Ex. 1. binary quadratic form [illegible]

$f = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$

[illegible] discriminant [illegible]

$D = a_0 a_2 - a_1^2$

[illegible] coeff. [illegible] rat. int. f [illegible]

$D' = a_0' a_2' - a_1'^2 = \Delta^2 D$

[illegible] D [illegible] invariant [illegible] index 2 [illegible]

$D = 0$ [illegible] quadr. f. [illegible] two pts [illegible] coincide [illegible]

II. [illegible] $K(a_0', a_1', \dots, a_n'; x_1', x_2') = \Delta^\lambda K(a_0, a_1, \dots, a_n; x_1, x_2)$

Coeff [illegible] rat. int. f. [illegible] x_1, x_2 [illegible] binary f. [illegible] K [illegible] f [illegible] ~~cov.~~ **covariant** [illegible] index λ [illegible] binary f. [illegible] pt syst. [illegible] pt syst. [illegible] proj. [illegible] invariant [illegible]

Ex. 2 $h = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}$ [illegible] f. **Hessian** [illegible]

$h' = \Delta^2 h$ [illegible]

Hessian index 2 [illegible] covariant [illegible]

[illegible] binary quadrat. form [illegible]

$h = 4(a_0 a_2 - a_1^2)$ [illegible]

cubic form $f = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3$

[illegible] $h = [(a_0 a_2 - a_1^2)x_1^2 + (a_0 a_3 - a_1 a_2) + (a_1 a_3 - a_2^2)x_2^2]$ [illegible]

III. [illegible] binary forms [illegible] system [illegible] simultaneous invariant

simult. covar. [illegible]

Ex. 3. $f_1 = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$

$f_2 = b_0 x_1^2 + 2b_1 x_1 x_2 + b_2 x_2^2$

$\theta = a_0 b_2 + a_2 b_0 - 2a_1 b_1$

[illegible] θ [illegible]

$\theta' = \delta^2 \theta$ [illegible]

[illegible] θ [illegible] f_1, f_2 [illegible] simult. inv. [illegible] index 2

$\theta = 0$ [illegible] $f_1 = 0, f_2 = 0$ [illegible] pt [illegible] harmonically separate [illegible] proj. property

Ex. 4. $J = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$ [illegible] $f_1 f_2$ [illegible] Jacobian

$J' = \delta J$ [illegible]

[illegible] Jacobian [illegible] index [illegible] covariant

$f_1 = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$

$f_2 = b_0 x_1^2 + b_2 x_2$

$J = 2[(a_0 b_1 - a_1 b_0)x_1 + (a_1 b_1 - a_2 b_0)x_2]$

$f_1' \quad f_2' \quad f_1'' \quad J'$

[illegible] harmonic [illegible] Jacobian [illegible] harmonic [illegible] proj. prop.

Ex. 5 [illegible] $f_1 = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$

$f_2 = b_0 x_1^2 + 2b_1 x_1 x_2 + a_2 x_2^2$

$J = (a_0 b_1 - a_1 b_0)x_1^2 + (a_0 b_2 - a_2 b_0)x_1 x_2 + (a_1 b_2 - a_2 b_1)x_2^2$

$\theta(f_1, J) \equiv 0$ $\quad \theta$ [illegible]

$\theta(f_2, J) \equiv 0$ identically

[illegible] $J' \quad f_1' \quad J'' \quad f_1'' \quad f_2' \quad f_2''$

J', J'' [illegible] $f_1' f_1''$ [illegible] $f_2' f_2''$ [illegible] harmonic separate [illegible]

[illegible] involution [illegible] double pt [illegible] pair [illegible]

J', J'' [illegible] $f_1 = 0$ $f_2 = 0$ [illegible] involution [illegible] double pts [illegible]

$J = 0$ [illegible] covariant

Ex. 3 [illegible] θ [illegible] resultant [illegible]

$f_1 f_2$ [illegible] n-th deg. binary f. [illegible] k [illegible] arbitrary constant [illegible] $f_1 + kf_2$ [illegible] n-th binary f.

$I(a_0 a_1 \ldots a_n)$ [illegible] f_1 [illegible] invariant [illegible]

$I(a_0' + kb_0', a_1' + kb_1', \ldots, a_n' + kb_n')$

$= \delta^{\lambda} I(a_0 + kb_0, a_1 + kb_1, \ldots, a_n + kb_n)$

[illegible] k [illegible] power [illegible] expand [illegible] equal power [illegible]
coeff [illegible] equate [illegible]

$$k \quad \frac{\partial I}{\partial a_0'} b_0' + \frac{\partial I}{\partial a_1'} b_1' + \cdots + \frac{\partial I}{\partial a_n'} = \Delta^\lambda \left(\frac{\partial I}{\partial a_0} b_0 + \cdots \frac{\partial I}{\partial a_n} b_n \right)$$

— — — —

$$[illegible] \quad \frac{\partial I}{\partial a_0} b_0 + \frac{\partial I}{\partial a_1} b_1 + \cdots + \frac{\partial I}{\partial a_n} b_n, \cdots$$

f, f_1 [illegible] simult. invariant [illegible]

f, f_1 quadr. [illegible] Θ [illegible]

[illegible] apolar [illegible] Θ [illegible] harmonic [illegible]

IV. $f(x_1, x_2)$ [illegible] pt (y_1, y_2) [illegible]

$$\begin{cases} x_1' = \alpha x_1 + \beta x_2 \\ x_2' = \gamma x_1 + \delta x_2 \end{cases} \qquad \begin{cases} y_1' = \alpha y_1 + \beta y_2 \\ y_2' = \gamma y_1 + \delta y_2 \end{cases}$$

[illegible] y [illegible] transf. [illegible]

$I(a_0, a_1, \cdots a_n; x_1, x_2; y_1, y_2)$ [illegible] constant factor [illegible]

[illegible] invariant [illegible] (mixed concomitant) [illegible]

[illegible] x_1, x_2 [illegible] contravariant [illegible]

Ex. 6 $\quad f = a_0 x_1^2 + 2 a_1 x_1 x_2 + a_2 x_2^2$

$$P = a_0 x_1 y_1 + a_1 (x_1 y_2 + x_2 y_1) + a_2 x_2 y_2$$

[illegible] mixed concomit. [illegible] (y_1, y_2) [illegible] $y =$

[illegible] polar [illegible] polar form [illegible]

[illegible] proj. property [illegible]

$P = 0$ [illegible] define [illegible] pt (x_1, x_2) [illegible]

$f = 0$ [illegible] define [illegible] pts [illegible] y [illegible] harmonic conj.

$f' \; P \; f'' \; y$ [illegible]

[illegible]

$f(x_1, x_2) = 0$ [illegible] x_1 [illegible] $l z_1 + m y_1, x_2$ [illegible] $l z_2 + m y_2$ [illegible]

$$f(l z_1 + m y_1, \; l z_2 + m y_2) = 0 \quad \frac{l}{m} \; [illegible]$$

$\frac{l}{m}$ [illegible] $y(y_1, y_2)$ [illegible] x [illegible] dist. [illegible] z [illegible] x [illegible] dist. [illegible]

ratio [illegible] $\quad z \overset{m}{—} x \overset{l}{—} y$

expand [illegible] $l^2 f(z_1, z_2) + 2 lm P(z_1, z_2; y_1, y_2) + m^2 f(y_1, y_2) = 0$

$$\frac{l^2}{m^2} + 2 \frac{l}{m} \frac{P}{f} + f = 0$$

coeff [illegible] $P = 0$ [illegible] roots [illegible] sum 0 [illegible]

$$\frac{y\overline{x'}}{z\overline{x'}} + \frac{y\overline{x''}}{z\overline{x''}} = 0 \qquad (1)$$

$$\frac{y\overline{x'}}{z\overline{x'}} : \frac{y\overline{x''}}{z\overline{x''}} = -1$$

[illegible] y, z [illegible] x', x'' [illegible] harmonic conj. [illegible]

(1) [illegible] $\quad y x' = y z + z x'$ [illegible]

$$= z x' - z y$$

$$\left(1 - \frac{zy}{zx'}\right)\left(1 - \frac{zy}{zx''}\right) = 0$$

$$\therefore \frac{2}{zy} = \frac{1}{zx'} + \frac{1}{zx''}$$ [illegible] harm. [illegible] def. [illegible]

[illegible] $f(x_1, x_2)$ [illegible] nth binary f [illegible]

$f(lz_1+my_1, lz_2+my_2) = l^n f(z_1, z_2) + \frac{n}{1} l^{n-1} m \left(y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2}\right) f(z_1 z_2)$

$+ \frac{n(n-1)}{2!} l^{n-2} m^2 \left(y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2}\right)^2 f(z_1, z_2) + \dots = 0$

$\left(y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2}\right) f(z_1, z_2) = 0$

[illegible] $(n-1)$ [illegible] deg. equat. [illegible] y [illegible]

1st polar [illegible]

$\left(y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2}\right)^2 f(z_1, z_2) = 0$ [illegible] 2nd polar [illegible]

1st polar [illegible] n roots [illegible]

$\frac{yx'}{zx'} + \frac{yx''}{zx''} + \dots + \frac{yx^{(n)}}{zx^{(n)}} = 0$ [illegible]

[illegible] $\frac{n}{zy} = \frac{1}{zx'} + \frac{1}{zx''} + \dots + \frac{1}{zx^{(n)}}$

[illegible] y [illegible] z [illegible] $x', x'' \dots x^{(n)}$ [illegible] harmonic centre [illegible]

Ex. 7. Cubic form [illegible]

$f = a_0x_1^3 + \dots$

z [illegible] second polar [illegible]

$(a_0y_1 + a_1y_2)z_1^2 + 2(a_1y_1 + a_2y_2)z_1z_2 + (a_2y_1 + a_3y_2)z_2^2$

[illegible] double [illegible]

$(a_0a_2 - a_1^2)y_1^2 + (a_0a_3 - a_1a_2)y_1y_2 + (a_1a_3 - a_2^2)y_2^2 = 0$

[illegible] cubic f [illegible] Hessian [illegible]

[illegible] cubic f [illegible] Hessian [illegible] equat. [illegible] pt. syst. [illegible]

[illegible] second polar [illegible] pt. double pt. [illegible]

V. [illegible] invar. I_1, I_2 [illegible] $I_1' = \Delta^{\lambda_1} I_1$,

$I_2' = \Delta^{\lambda_2} I_2$ [illegible]

$\frac{I_1'^{\lambda_2}}{I_2'^{\lambda_1}} = \frac{\Delta^{\lambda_1\lambda_2} I_1^{\lambda_2}}{\Delta^{\lambda_1\lambda_2} I_2^{\lambda_1}} = \frac{I_1^{\lambda_2}}{I_2^{\lambda_1}}$

[illegible] $\frac{I_1^{\lambda_2}}{I_2^{\lambda_1}}$ absolute invariant [illegible]

Ex. 8. four pts. [illegible] anharmonic ratio absolute inv. [illegible]

[illegible] Theory of invariant. Inv. covariant, mixed concomit. [illegible] algebra [illegible] Binary form [illegible] pt. rank [illegible] elem. f. [illegible] proj. geom. [illegible] proj. geom. [illegible] expositi. [illegible] Invar. theory [illegible] Cayley 1854 [illegible] Boole [illegible] motion [illegible] invar. [illegible] Sylvester, Salmon [illegible] Cayley Sylvester [illegible] Salmon [illegible] Cayley [illegible] Aronhold [illegible]

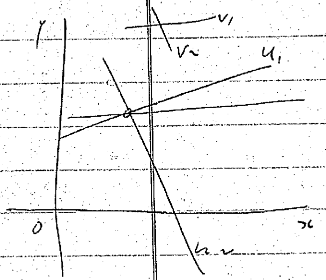
明ヒ Clebsch (1860年) [illegible] Gordan [illegible]
Hermite [illegible] Brioschi [illegible] Clebsch-
Lindemann Vorl. d. Geom. 1891 [illegible] Clebsch [illegible] synth.
geom. algebraic geom. diff. geom. [illegible] algebraic
geom. [illegible] Invar. [illegible] Abelian [illegible]
[illegible] surface [illegible] systematic study
[illegible] (diff. g. [illegible] Darboux,
alg. g. [illegible] Castelnuovo, Enriques [illegible])

58. Projective pencils. Circular points at infinity.

I. [illegible] plane [illegible] rectangular coord. [illegible] x, y [illegible]

$u_1 \equiv a_1 x + b_1 y + c_1 = 0$

$u_2 \equiv a_2 x + b_2 y + c_2 = 0$



k [illegible] arbitrary const. [illegible]

$u_1 + k u_2 = 0$ [illegible] u_1, u_2 [illegible]

[illegible] vertex [illegible] flat pencil [illegible]

$k = k_1, k_2, k_3, k_4$ [illegible]

4 values [illegible] anharmonic ratio [illegible]

$$\frac{k_1 - k_3}{k_2 - k_3} : \frac{k_1 - k_4}{k_2 - k_4}$$

$\left(\begin{matrix} u_1 + k u_2 = 0 \\ v_1 + k v_2 = 0 \end{matrix}\right.$ Plücker analyt. notation [illegible] (anharm. [illegible])

[illegible] coor. el. [illegible] k [illegible] elimination

$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 0 \quad \therefore \quad u_1 v_2 - u_2 v_1 = 0$

[illegible] 2nd deg. curve [illegible] Steiner [illegible] 2nd ord. curve [illegible] generation [illegible]

[illegible] space [illegible] rectangular coord. x, y, z [illegible]

$u_1 \equiv a_1 x + b_1 y + c_1 z + d_1 = 0$ plane,

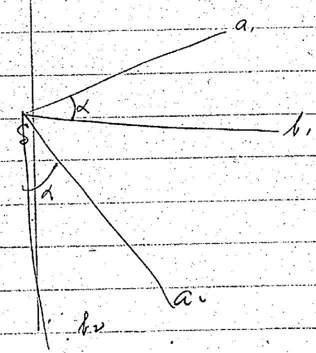
$u_1 + k u_2 = 0$ [illegible] axial pencil [illegible]

$v_1 + k v_2 = 0$ [illegible] proj. [illegible] axial p. [illegible]

[illegible] coor. pl. [illegible] $u_1 v_2 - v_1 u_2 = 0$ [illegible]

2nd deg. ruled surf. [illegible]

II. [illegible] plane [illegible] angle α [illegible] vertex [illegible] rotation [illegible] one pair [illegible] proj. pencil [illegible]

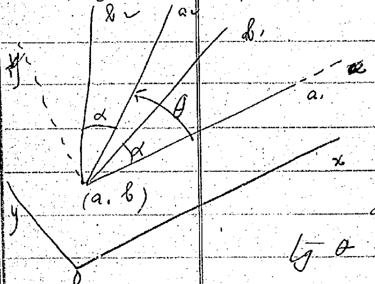


$a_1\ a_2\ a_3 \cdots$

$b_1\ b_2\ b_3 \cdots$ [illegible]

$$\therefore \frac{\sin(a_1 a_3)}{\sin(a_2 a_3)} : \frac{\sin(a_1 a_4)}{\sin(a_2 a_4)} = \frac{\sin(b_1 b_3)}{\sin(b_2 b_3)} : \frac{\sin(b_1 b_4)}{\sin(b_2 b_4)}$$ [illegible]

[illegible] proj. pencil [illegible] double line [illegible] imag. [illegible]
ab [illegible] double line [illegible] eq. [illegible]



α constant angle θ variable

$y-b=(x-a)\tan\theta$ a's eq.

$y-b=(x-a)\tan(\theta+\alpha)$ b's

double line [illegible]

$\tan\theta=\tan(\theta+\alpha)$ [illegible]

$$=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$\tan\theta-\tan^2\theta\tan\alpha=\tan\theta+\tan\alpha$

$\tan^2\theta=-1$

$\tan\theta=\pm i$

double l. [illegible] $y-b=\pm i(x-a)$

$x\pm iy=const.$ (vertex [illegible] depend [illegible])

Theorem [illegible] double line [illegible] α [illegible]

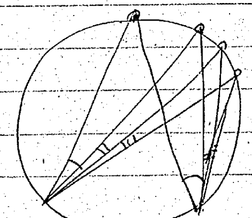
Def. [illegible] double l. [illegible] isotropic line (Laguerre) [illegible]

[illegible] minimal l. (Lie) [illegible] minimal l [illegible]

$(x-a)^2+(y-b)^2=0$ [illegible] line Poncelet

1822 [illegible]

III circle [illegible] 関係 [illegible]



[illegible] pencil [illegible] Steiner theorem [illegible] particular case [illegible] circle [illegible] 2nd ord. [illegible] circle [illegible] base [illegible] curve [illegible] 2nd ord. curve [illegible]

2nd ord. [illegible] curve [illegible] plane sect. [illegible]

circle [illegible] eq. $(x-a)^2+(y-b)^2-r^2=0$

$\{(x-a)+i(y-b)\}\{(x-a)-i(y-b)\}-r^2=0$

$$\begin{cases}[(x-a)+i(y-b)]+\lambda r=0\\ r+\lambda[(x-a)-i(y-b)]=0\end{cases}$$

[illegible] eq. [illegible] λ [illegible]

[illegible] plane [illegible] pt. [illegible] homog. coord. [illegible]

$x=\frac{x_1}{z_1}$, $y_1=\frac{y_1}{z_1}$ [illegible]

(x_1, y_1, z_1) [illegible] homog. coord. [illegible]

[illegible] x_1, y_1, z_1 [illegible] finite [illegible]

$z_1=0$ [illegible] pt. at ∞ [illegible]

[illegible] $z_1=0$ [illegible] line at ∞ eq. [illegible]

$$\begin{cases}[(x_1-az_1)+i(y_1-bz_1)]+\lambda r z_1=0\\ rz_1+\lambda[(x_1-az_1)-i(y_1-bz_1)]=0\end{cases}$$

[illegible] $(x_1-az_1)+i(y_1-bz_1)=0$

$rz_1=0$

[illegible] flat pencil [illegible]

$z_1 = 0$ line at ∞

$(x_1 - az_1) + i(y_1 - bz_1) = 0$ [illegible]

$(x-a) + i(y-b) = 0$ [illegible] a b [illegible]

min. line [illegible]

[illegible]

$z_1 = 0$, $(x-a) - i(y-b) = 0$ [illegible]

[illegible] line at ∞ [illegible] a b [illegible] min. l. [illegible]

[illegible] flat pencil [illegible] pencil [illegible]

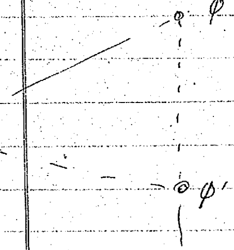
[illegible] proj. [illegible] corr. l. [illegible] circle [illegible]

[illegible] circle [illegible] pencil [illegible] vertex [illegible]

[illegible] circle [illegible] imag. pt [illegible]

$(x-a) + i(y-b) = 0$ [illegible] $x + iy = 0$ [illegible] // [illegible]

[illegible] $(x-a) - i(y-b) = 0$ [illegible] $x - iy = 0$ [illegible]



φ, φ' [illegible] line at ∞ [illegible] fix [illegible] imag. pt [illegible] a, b [illegible]

Theor. [illegible] pla [illegible] circle [illegible] fixed pts φ, φ' [illegible] circular pt at ∞ [illegible]

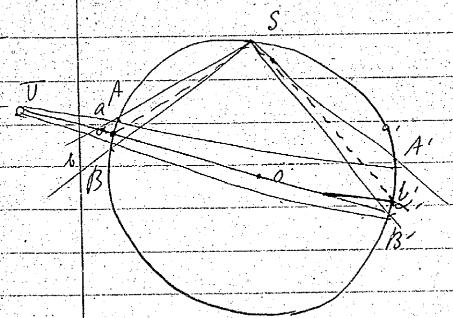
Poncelet 18[illegible] conic [illegible] circle [illegible]

Lemma [illegible] involutional pencil [illegible]

[illegible] corr. l. [illegible] pair [illegible]

[illegible] pair [illegible]

Proof. invol. p[illegible]S [illegible]



§35 [illegible]

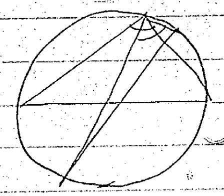
AA', BB' [illegible] invol.

Centre U [illegible]

∠S, ∠'S [illegible] corr. l. [illegible]

[illegible] U [illegible] O [illegible]

[illegible] pair [illegible]



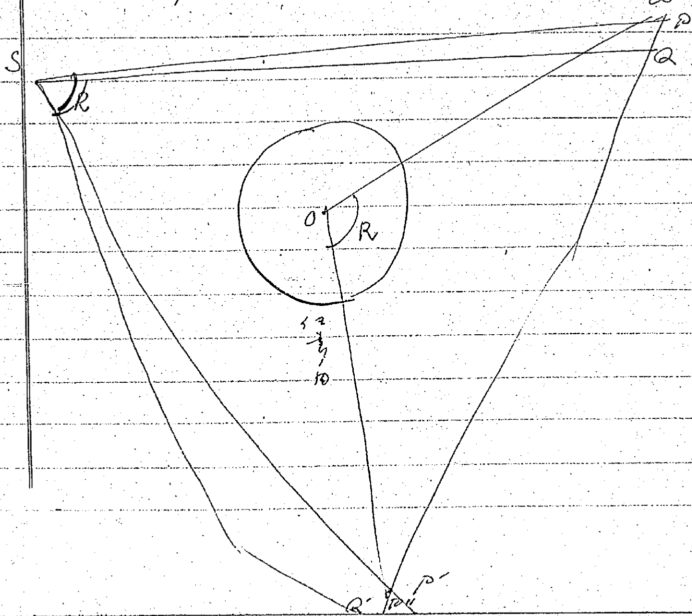
Def. [illegible] __Circular involution__ [illegible]

Def. Circ. invol. の double line [illegible] imag. [illegible] __Minimal line__ [illegible]

Invol. [illegible] two pair [illegible] cor. l. [illegible] min. l. [illegible] l. at ∞ [illegible] plane [illegible] fixed pt [illegible] vertex [illegible]

Def. [illegible] circ. invol. [illegible] l. at ∞ [illegible] pt range [illegible] invol. [illegible] double pt. [illegible] plane [illegible] fixed pt [illegible] __circular pts at ∞__ [illegible]

[illegible] circ. pt at ∞ [illegible] pair [illegible]

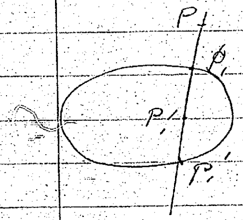


circle [illegible] P [illegible] polar [illegible] O [illegible]

POP'' [illegible] R [illegible] P ∞ [illegible] $OP \parallel SP$, $SP' \parallel SP''$

P [illegible] polar [illegible] P' [illegible] P' [illegible] polar [illegible] P [illegible]

PP' [illegible] circle [illegible] conj. pts [illegible] circle [illegible] line at ∞ [illegible] real pt [illegible] imag. pt [illegible] (Staudt theorem) [illegible] Φ, Φ' [illegible]



PP' [illegible] QQ' [illegible] ϕ, ϕ' [illegible] harm. [illegible] PP', QQ' [illegible] invol. [illegible] double pts [illegible] circular invol. [illegible] circ. pt at ∞ [illegible]

ϕ, ϕ' [illegible] circle [illegible] l. at ∞ [illegible] circle [illegible] ϕ, ϕ' [illegible]

__Corol.__ [illegible] SP, SP' [illegible] min. l. $S\Phi$, $S\Phi'$ [illegible] harm. [illegible] circular pt at ∞ [illegible] projection [illegible] definition [illegible]

__Axis & foci of the conic__ [illegible] conic [illegible]

S [illegible] vertex [illegible] conj. line aa', bb' [illegible] (aa'), (bb') [illegible] invol. [illegible] $(a, b, c, \ldots)$ $(a', b', c', \ldots)$ [illegible] a [illegible] a' [illegible] corresp. [illegible]

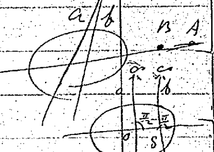
[illegible] a' [illegible] a [illegible] double line [illegible] S [illegible] tangent [illegible] invol. [illegible] ~~invol. double line~~ [illegible] one pair [illegible]

Def. [illegible] conj. diameter [illegible] axis [illegible] circular involution [illegible] it [illegible] ~~circle~~ ellipse circle [illegible] ~~conj~~ conj. involution double line [illegible] hyperbola [illegible] asympt. ellipse [illegible] imag. line, parabola [illegible] line at ∞, circle [illegible] minimal line [illegible]

Remark [illegible] axis def. [illegible] $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ [illegible] Bonchi Comb. II. 3. §1144–47. Reye [illegible]

Theorem 1. [illegible] diameter [illegible] asympt. [illegible] diam. [illegible] conj. line [illegible] diam. [illegible] diam. [illegible] axis [illegible]

[illegible] axis [illegible] asympt [illegible] axis [illegible] conj. line [illegible] axes [illegible] ⊥ [illegible]



Proof. a, b [illegible] C [illegible] polar [illegible] c [illegible]

a [illegible] pole A, b [illegible] B [illegible] AB [illegible] c [illegible]

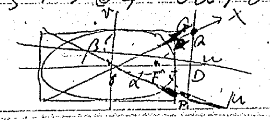
[illegible] c [illegible] a, b [illegible] conj. [illegible]

[illegible] c [illegible] diam. [illegible] b [illegible] c [illegible] conj. [illegible] b [illegible] pt. at ∞ [illegible] polar [illegible] c [illegible]

centre O [illegible] c [illegible] ⊥ a [illegible] a, b [illegible] ∥ [illegible] pt. at ∞ [illegible] a, b conj. [illegible] c [illegible] axis [illegible]

Def. conic [illegible] plane [illegible] F [illegible] all lines [illegible] line conj. l [illegible] ⊥ [illegible] F [illegible] Conic Focus [illegible] analog. [illegible] theorem [illegible] Def [illegible] F [illegible] diam. [illegible] O centre, [illegible] diam. theorem 1 [illegible] conic axis [illegible]

[illegible] F [illegible] $O\bar{u}$, $O\bar{v}$ = ∥ [illegible] pair [illegible] conj. diam. [illegible]

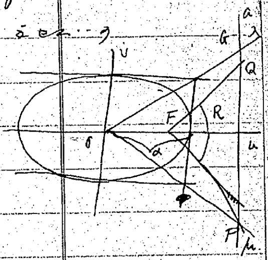


[illegible] on pair [illegible] F [illegible] focus [illegible] F [illegible] vertex [illegible] pair [illegible] conj. l [illegible] ⊥ [illegible] conj. l [illegible] circular involution [illegible] corresp. [illegible] ⊥ [illegible] F [illegible] polar [illegible] at ∞ [illegible] ∥ [illegible] rectangle [illegible] $O\bar{\lambda}$, $O\bar{\mu}$ [illegible] conj. diam. [illegible] F [illegible] OG = ∥ [illegible] $\bar{FQ}$ [illegible] F [illegible] polar GDP [illegible] FQ [illegible] pt. at ∞ [illegible] polar [illegible] $O\bar{\lambda}$ [illegible] pt. at ∞ [illegible] polar [illegible] $O\bar{\mu}$ [illegible] FQ [illegible] pole [illegible] P [illegible] FQ ⊥ FP [illegible] conj. [illegible] F focus [illegible] necess. + suff. cond. [illegible] ∠QFP = R [illegible] $\bar{FD}^2 = PD \cdot DQ$ [illegible]

$$\bar{FD}^2 = DG \cdot DQ \qquad \frac{DQ}{FD} = \frac{DG}{OD} = \frac{\beta}{\alpha}$$

$$\frac{DQ \cdot DG}{FD \cdot OD} = \frac{\beta^2}{\alpha^2}$$

$$\frac{FD}{OD} = \frac{\beta^2}{\alpha^2} \quad \therefore \frac{OF}{OD} = \frac{\alpha^2-\beta^2}{\alpha^2}$$



GP [illegible] F [illegible] polar [illegible] $OF \cdot OD = \alpha^2$

$\bar{OF}^2 = \alpha^2 - \beta^2 \qquad OF = \pm\sqrt{\alpha^2-\beta^2}$ [illegible] F major axis [illegible] $\alpha > \beta$, F [illegible] real minor axis [illegible] $\alpha < \beta$, F [illegible] imag. pt. [illegible]

Theorem 2. [illegible] focus [illegible] conic [illegible] imag. tangent [illegible] minimal l. [illegible] (F [illegible] conic [illegible] tang. [illegible])

[illegible] F [illegible] circular involution [illegible] double line [illegible] F [illegible] conj. l. [illegible] tang. [illegible] circular involution [illegible] double l. [illegible] minimal l. [illegible]

Theorem 3. [illegible] circular pts at [illegible] ~~[illegible]~~ 2 pair of tangents [illegible] [illegible] mag. [illegible] [illegible] axis [illegible] focus [illegible] ~~[illegible]~~ real [illegible] mag. [illegible] (i.e. [illegible] 2 [illegible])

[illegible] (Poncelet 1822?). [illegible] Plücker 1832 [illegible] def [illegible] circular pts at [illegible] any algebraic curve [illegible] tangents [illegible] focus [illegible]

Cor. class n [illegible] curve [illegible] focus [illegible] circular pt at [illegible] n [illegible] focus [illegible] n^2 [illegible]

Chapter XIV. Projective transformations of the plane

60. Collineation

Theorem 1. plane field [illegible] collin. [illegible] homog. coord. $(x_1 x_2 x_3)$ [illegible] linear transf.

$$x_1' = a_1x_1 + b_1x_2 + c_1x_3$$
$$x_2' = a_2x_1 + b_2x_2 + c_2x_3 \qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$
$$x_3' = a_3x_1 + b_3x_2 + c_3x_3$$

[illegible] non-homog. coord. xy [illegible]

$$x' = \frac{a_1x_1 + b_1y_0 + c_1}{a_3x + b_3y_0 + c_3}$$

$$y' = \frac{a_2x_0 + b_2y + c_2}{a_3x + b_3y + c_3} \quad [illegible]$$

Proof. E [illegible] plane [illegible] $(x_1 x_2 x_3)$ [illegible] E' [illegible] $(x_1' x_2' x_3')$ [illegible] $x_1' = \phi(x_1 x_2 x_3)$, $x_2' = \chi(\;)$, $x_3' = \psi(\;)$

ϕ, χ, ψ [illegible] one valued f. [illegible]

[illegible] E' [illegible] $l_1'x_1' + l_2'x_2' + l_3'x_3' = 0$ [illegible]

E [illegible] curve $l_1'\phi(x_1 x_2 x_3) + l_2'\chi(\;) + l_3'\psi(\;) = 0$ [illegible]

[illegible] curve [illegible] l_1', l_2', l_3' [illegible]

$$x_1' = a_1x_1 + b_1x_2 + c_1x_3$$
$$x_2' = a_2x_1 + b_2x_2 + c_2x_3$$
$$x_3' = a_3x_1 + b_3x_2 + c_3x_3 \quad [illegible]$$

[illegible] E' [illegible] one pt $(x_1' x_2' x_3')$ [illegible] E [illegible] one pt $(x_1 x_2 x_3)$ [illegible]

[illegible] linear eq. [illegible] $x_1 x_2 x_3$ [illegible] solu [illegible]

[illegible] solution [illegible]

$$\Delta \neq 0, \quad x_1' = a_1x_1 + b_1x_2 + c_1x_3$$
$$x_2' = \ldots$$
$$x_3' = \ldots$$

Cor. plane [illegible] collineation [illegible] coeff. [illegible] ratio [illegible] plane [illegible] collineation [illegible] 8-terms [illegible] continuous group [illegible] (x y) pt [illegible] invariant pt [illegible]

$$x' = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3} \qquad y' = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}$$

[illegible] ρ [illegible]

$$\begin{cases} \rho x = a_1 x + b_1 y + c_1 \\ \rho y = a_2 x + b_2 y + c_2 \\ \rho = a_3 x + b_3 y + c_3 \end{cases} \qquad \begin{cases} (a_1 - \rho) x + b_1 y + c_1 = 0 \\ a_2 x + (b_2 - \rho) y + c_2 = 0 \\ a_3 x + b_3 y + (c_3 - \rho) = 0 \end{cases}$$

[illegible] linear eq [illegible]

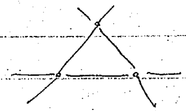
$$\begin{vmatrix} a_1 - \rho & b_1 & c_1 \\ a_2 & b_2 - \rho & c_2 \\ a_3 & b_3 & c_3 - \rho \end{vmatrix} = 0$$

[illegible] ρ [illegible] cubic eq [illegible] 3 roots [illegible] real [illegible] ρ value [illegible] eq [illegible] α [illegible] $x\,y$ [illegible]

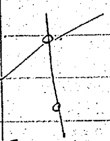
Theorem [illegible] general collineation [illegible] invar. pt [illegible] real [illegible] invariant pts [illegible] invariant line [illegible]

Remark. [illegible] inv. pt [illegible] inv. line [illegible] coincide [illegible] ρ [illegible] cubic eq [illegible] quadratic or linear eq [illegible] reduce [illegible]

Type. I.



II

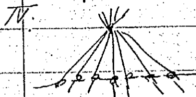


inv. pt [illegible]

III

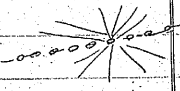
inv. pt [illegible]

IV.



inv. pt co[illegible]

V



[illegible] inv. pt or line [illegible] real or imag. [illegible] finite or infinite [illegible] collineation [illegible] systematic [illegible] classify [illegible] (Lie, Vorlesungen ü. Continuierliche Gruppen (1893). [illegible] Meyer, Chicago Congress Papers 1898, Newson, Amer. J. M. 24 1902)

Binary form [illegible] inv. theory [illegible] line [illegible] proj. gen. [illegible] 解 [illegible] ternary form $f(x_1 x_2 x_3)$ [illegible] inv. th. [illegible] plane [illegible] proj. gen. [illegible] ternary quadratic form [illegible] inv. th. [illegible] conic section [illegible] proj. gen. [illegible] cubic form [illegible] inv. th. [illegible] cubic curve [illegible] proj. gen. [illegible] cubic quartic [illegible] inv. th. [illegible] Salmon [illegible] ternary form [illegible] inv. th. [illegible] Clebsch-Lindemann, Vorl. d. G. I. 1. Andoyer, Leçons sur la théorie des formes, t. 1 [illegible]

61. *Infinitesimal proj. transformations* (proj. [illegible] collineation [illegible])

[illegible] plane [illegible]

$$x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3} \qquad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}$$

identical [illegible]

$a_1 = 1 \qquad b_1 = 0 \qquad c_1 = 0$

$a_2 = 0 \qquad b_2 = 1 \qquad c_2 = 0$

$a_3 = 0 \qquad b_3 = 0 \qquad c_3 = 1$

[illegible] coeff. [illegible] infinit. [illegible]

$a_1 = 1 + \alpha_1 \delta t \qquad b_1 = \beta_1 \delta t \qquad c_1 = \gamma_1 \delta t$

$a_2 = \alpha_2 \delta t \qquad b_2 = 1 + \beta_2 \delta t \qquad c_2 = \gamma_2 \delta t$

$a_3 = \alpha_3 \delta t \qquad b_3 = \beta_3 \delta t \qquad c_3 = 1 + \gamma_3 \delta t$

α, β, γ arbitrary const. [illegible] δt [illegible] infinit. quantity [illegible]

$$x_1 = \frac{x + (\alpha_1 x + \beta_1 y + \gamma_1)\delta t}{1 + (\alpha_3 x + \beta_3 y + \gamma_3)\delta t}, \quad y_1 = \frac{y + (\alpha_2 x + \beta_2 y + \gamma_2)\delta t}{1 + (\alpha_3 x + \beta_3 y + \gamma_3)\delta t}$$

[illegible] δt [illegible] power series [illegible] expand [illegible]

$$x_1 = x + [\gamma_1 + (\alpha_1 - \gamma_3)x + \beta_1 y - \alpha_3 x^2 - \beta_3 xy]\delta t + \cdots$$

$$y_1 = y + [\gamma_2 + \alpha_2 x + (\beta_2 - \gamma_3)y - \alpha_3 xy - \beta_3 y^2]\delta t + \cdots$$

$(\delta t)^2$ [illegible]

$$\delta x = x_1 - x = (a + cx + dy + hx^2 + kxy)\delta t,$$

$$\delta y = y_1 - y = (b + ex + gy + hxy + ky^2)\delta t$$

[illegible] $f(x,y)$ [illegible] continuity, differentiability [illegible] permutability [illegible]

$$\delta f = f(x_1 y_1) - f(xy) = f(x + \delta x, y + \delta y) - f(xy)$$

$$= f(xy) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y - f(xy)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$\delta x, \delta y$ [illegible]

$$\delta f = \left\{(a + cx + dy + \cdots)\frac{\partial f}{\partial x} + (b + ex + \cdots)\frac{\partial f}{\partial y}\right\}\delta t$$

[illegible]

$$Uf \equiv \left\{(a + cx + dy + hx^2 + kxy)\frac{\partial f}{\partial x} + (b + ex + gy + hxy + ky^2)\frac{\partial f}{\partial y}\right\}$$

[illegible]

Uf [illegible] *infinites. proj. tr.* [illegible] symbol [illegible]

$Ux = a + cx + dy + hx^2 + kxy$

$Uy = b + ex + gy + hxy + ky^2$

Uf [illegible] constant $a\ b\ c\ d\ e\ g\ h\ k$ [illegible] general [illegible] inf. proj. [illegible] 8 special inf. proj. tr.

$p,\ q;\ xp,\ yp,\ xq,\ yq;\ x^2p + xyq,\ xyp + y^2q$

[illegible] combin. [illegible]

Ex. 1 $\quad Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ [illegible] collineat. [illegible]

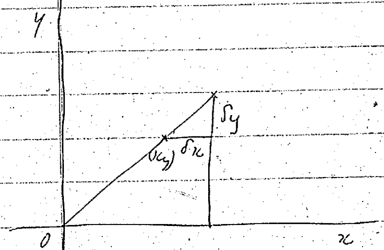
[illegible] $\delta x = (a + cx + \cdots)\delta t = Ux\,\delta t$ [illegible]

$\delta x = x\,\delta t$, [illegible] $\delta y = Uy\,\delta t = y\,\delta t$

$\frac{\delta y}{\delta x} = \frac{y}{x}$

$\sqrt{(\delta x)^2+(\delta y)^2} = \sqrt{x^2+y^2}\, dt$

orig. ... dist. co... vary ... infinitesimal similar transf. ...

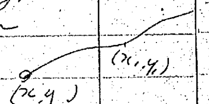


§3 ... t ... time ... t ... dt ... vary ... xy ... posit ... $\delta x, \delta y$ ... vary ... current coord. x_1, y_1 ... t ... funct ...

$dx_1 = (a + cx_1 + dy_1 + hx_1^2 + kx_1y_1)\,dt$

$dy_1 = (b + ex_1 + gy_1 + hx_1y_1 + ky_1^2)\,dt$

... $t=0$ ... $x_1 = x,\ y_1 = y$ ...



(1) $\frac{dx_1}{a_1 + cx_1 + dy_1 + hx_1^2 + kx_1y_1} = \frac{dy_1}{b + ex_1 + gy_1 + hx_1y_1 + ky_1^2} = dt$

when $x_1 = x,\ y_1 = y$ for $t=0$ ... diff. eq. ... sol. ... inf. ... t ... continuous vary ... co' ... inf. ... transf. ... group ...

... inf. ... one parameter group ...
(t ... co')

Ex. 2 $\quad Uf \equiv -yp + xq$

$\delta x = -y\,\delta t, \quad \delta y = x\,\delta t$

$\frac{dx_1}{-y_1} = \frac{dy_1}{x_1} = dt$

$x_1dx_1 + y_1dy_1 = 0$

$\therefore x_1^2 + y_1^2 = k^2$ (const.)

$y_1 = \sqrt{k^2 - x_1^2}$

$\frac{dx_1}{-y_1} = dt$ ... $-\frac{dx_1}{\sqrt{k^2-x_1^2}} = dt$

$\arccos \frac{x_1}{k} - t = c$ (const)

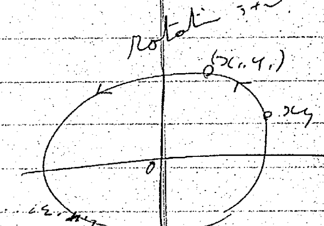
$\frac{x_1}{k} = \cos(c-t)$

$\frac{y_1}{k} = \sin(c-t)$

$(x_1)_{t=0} = x = k\cos c$

$(y_1)_{t=0} = y = k\sin c$

$\therefore \begin{cases} x_1 = x\cos t - y\sin t \\ y_1 = x\sin t - y\cos t \end{cases}$



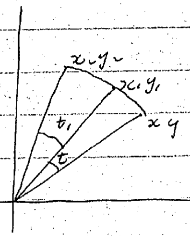
... one parameter group ...

$\begin{cases} x_2 = x_1\cos t_1 - y_1\sin t_1 \\ y_2 = x_1\sin t_1 + y_1\cos t_1 \end{cases}$

$x_2 = (x\cos t - y\sin t)\cos t_1 - (x\sin t + y\cos t)\sin t_1$

$= x(\cos t\cos t_1 - \sin t\sin t_1) - y(\sin t\cos t_1 + \cos t\sin t_1)$

$$\begin{cases} x_2 = x\cos(t+t_1) - y\sin(t+t_1) \\ y_2 = x\sin(t+t_1) + y\cos(t+t_1) \end{cases}$$



Ex. 3. $Uf = xp + yq$ inf. similar transf. [illegible]

$\delta x = x\delta t$, $\delta y = y\delta t$

$\frac{dx_1}{x_1} = \frac{dy_1}{y_1} = dt$ $x_1 = x$, $y_1 = y$ for $t = 0$

$x_1 = xe^t$, $y_1 = ye^t$

$x_2 = x_1 e^{t_1}$, $y_2 = y_1 e^{t_1}$

$x_2 = xe^{t+t_1}$, $y_2 = ye^{t+t_1}$

Theorem 2. Inf. proj. tr. [illegible] one parameter group proj. transf. [illegible]

finite proj. tr. [illegible] inf. proj. tr. [illegible]

[illegible] group [illegible] finite proj. tr. [illegible]

$\frac{dx_1}{a+ex_1+\cdots} = \frac{dy_1}{b+ex_1+\cdots} = dt$, $x_1 = x$, $y_1 = y$ for $t=0$

[illegible] solution: $x_1 = \frac{a_1x+b_1y+c_1}{a_3x+b_3y+c_3}$, $y_1 = \frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}$

a, b, c arbitrary funct. of t [illegible]

[illegible] Lie Cont. Gr. 41–44

Lie, Diff. gleich. Kap. 16 § 3 [illegible]

62 Invariant points, invariant curves, path curves.

Inf. tr. $Uf = \xi p + \eta q$ [illegible]

$\delta x = \xi \delta t$ $\delta y = \eta \delta t$

[illegible] $\xi = 0$, $\eta = 0$ [illegible] pt. [illegible] invariant pt. [illegible] $\delta = 0$ [illegible] variation [illegible]

[illegible] eq. [illegible] curve [illegible] pt. [illegible] inv. pt.

$\frac{\delta x}{\xi} = \frac{\delta y}{\eta}$ [illegible] xy pt. [illegible]

path curve (Bahnkurve) [illegible] $\frac{dy}{dx} = \frac{\eta}{\xi}$

[illegible] sol. [illegible] ∞^1 [illegible] curves

path curve [illegible] invariant curve [illegible]

[illegible] curve [illegible] pt. [illegible]

[illegible] curve [illegible] pt. [illegible] inv. c.

[illegible] pt. [illegible] inv. c.

[illegible]

Ex. 1. $Uf = xp$

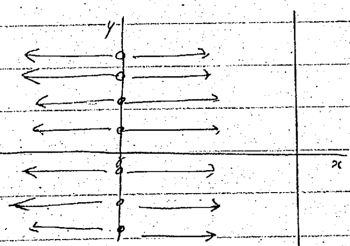
$\delta x = x\delta t$, $\delta y = 0\cdot\delta t$

$\frac{dx_1}{x_1} = \frac{dy_1}{0} = dt$ $x_1 = x$, $y_1 = y$ for $t = 0$

$x_1 = x e^t \qquad y_1 = y$

y axis ... inv. curve ...

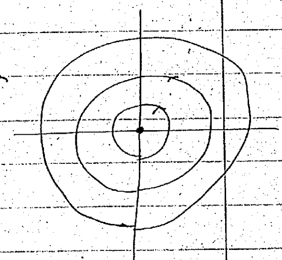
y = const ... path curve ...



Ex. 2 $Uf = -yp + xq$ ... rotation

origin ... invar. pt.

concentric circles, path curve ...



$-y = 0,\ x = 0$ ... origin

$$\frac{dx}{-y} = \frac{dy}{x}$$

$$x\,dx + y\,dy = 0$$

$$\therefore \quad x^2 + y^2 = \text{const.}$$

... Lie ... theory ... co ...

... path curve

$$\frac{dx}{a + ex + dy + hx^2 + kxy} = \frac{dy}{b + ex + gy + hxy + ky^2}$$

... Diff eq. ... Jacobi's dif. eq. ...

... sol. Serret Scheffers. Dif. u. Int. R. III. 227 — 248.

... Cromact, Analyse II p. 331 (Darboux's method)

... geom. consideration ...

... coord. axis ... inv. line ...

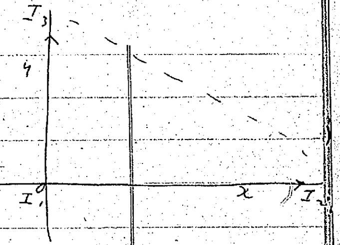
x, y axis, line at co ...

$\delta x = (a + cx + dy + hx^2 + kxy)\,\delta t$

$\delta y = (b + ex + gy + hxy + ky^2)\,\delta t$

$x = 0,\ y = 0$ ... inv. pt.

$\therefore \quad a = 0,\ b = 0.$



I_3 ... pt ... inv. ...

$x = \text{const.}$ ... $x + \delta x = \text{const.}$ ... parallel ...

$\therefore \quad \delta x$ ... y ... independent ...

$\therefore \quad d = 0,\ k = 0.$

... I_1 ... inv. ... $e = 0,\ h = 0.$

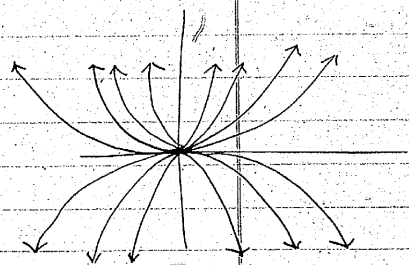
$$\therefore \begin{cases} \delta x = cx\,\delta t \\ \delta y = gy\,\delta t \end{cases}$$

$$\therefore \quad Uf = cx\frac{\partial f}{\partial x} + gy\frac{\partial f}{\partial y}$$

$$\frac{dx}{cx} = \frac{dy}{gy}$$

$$\frac{1}{c}\log x = \frac{1}{g}\log y + \log k_1$$

$$x^c y^{-g} = k_1$$



... homog. coord. ...

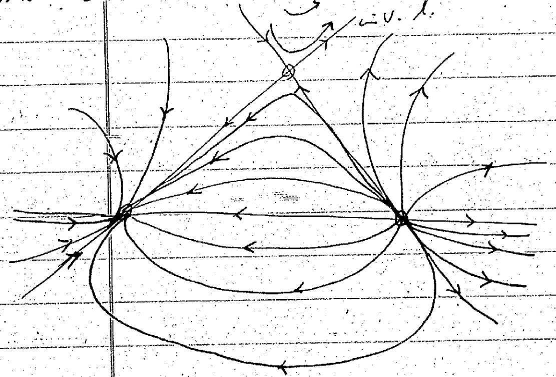
$$\frac{x_1}{x_3} = x, \quad \frac{x_2}{x_3} = y$$

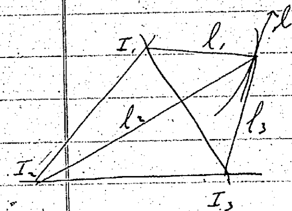
$x_1^{c} x_2^{-g} x_3^{-c+g} = k$

$x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} = k, \qquad \lambda_1+\lambda_2+\lambda_3 = 0.$

[illegible] : $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ [illegible]

path curve $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} = k$ [illegible]

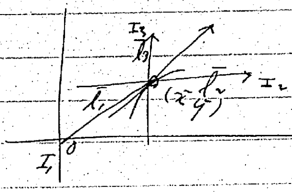




l tangent

$R_l\,(l, l_1, l_2, l_3)$

[illegible] special case [illegible]



$\bar{l} : y - \bar{y} = \frac{d\bar y}{d\bar x}(x-\bar x) = \frac{g\bar y}{c\bar x}(x - \bar x)$

$\bar{l}_1 : y - \bar y = \frac{\bar y}{\bar x}(x - \bar x)$

$\bar{l}_2 : y - \bar y = 0\,(x - \bar x)$

$\bar{l}_3 : y - \bar y = \infty\,(x - \bar x)$

$R_{\bar l} = \frac{\infty - \frac{\bar y}{\bar x}}{\infty - 0} : \frac{\frac{g\bar y}{c\bar x} - \frac{\bar y}{\bar x}}{\frac{g\bar y}{c\bar x} - 0} = \frac{g}{g-c}$

[illegible] path [illegible] pt M [illegible] l, MI_1, MI_2, MI_3 double ratio [illegible] Klein, Lie [illegible] 1870 [illegible] W-curve [illegible] v. Staudt [illegible] constant [illegible] form [illegible] 4 elements [illegible] Wurf [illegible] space [illegible] W-surface [illegible] diff. surf. [illegible] W-surface [illegible] Weingarten's surface [illegible] Klein, Lie [illegible] 1870 [illegible] Paris [illegible] Jordan [illegible] mot. [illegible] group [illegible] Darboux [illegible] Paris [illegible] mot. group [illegible] Lie [illegible] 1870–74 [illegible] continuous group [illegible] Klein [illegible] discont. group [illegible] automorph [illegible] 1883 Halphen [illegible] curve [illegible] Anharmonic curve [illegible]

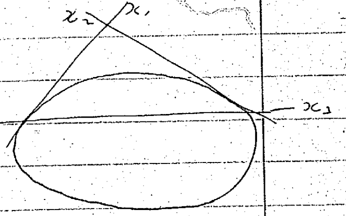
[illegible] eq. $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} = k \qquad (\lambda_1+\lambda_2+\lambda_3=0)$

[illegible] particular case [illegible]

1. $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = -2$

$x_1 x_2 = k x_3^2$

is conic [illegible]



II. $\frac{x_1}{x_3} = x + iy$

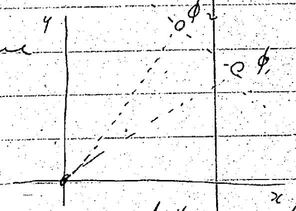
$\frac{x_2}{x_3} = x - iy$

xy rectangular coord.

$x_1 = 0,\ x_2 = 0$: minimal line

$x_3 = 0$ line at ∞

O (ϕ, ϕ') (circular pt at ∞) [illegible]



invariant [illegible] curve [illegible]

$x_1^{\frac{1+\lambda i}{2}} x_2^{\frac{1-\lambda i}{2}} x_3^{-1} = k$ (λ arbitrary const.)

is W-curve

$(x+iy)^{\frac{1+\lambda i}{2}} (x-iy)^{\frac{1-\lambda i}{2}} = k$

$(x^2+y^2)^{\frac{1}{2}} \left(\frac{x+iy}{x-iy}\right)^{\frac{\lambda i}{2}} = k$

[illegible] polar coord. [illegible] $x = r\cos\theta,\ y = r\sin\theta$

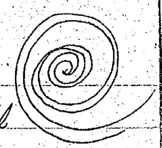
$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

$\frac{x+iy}{x-iy} = e^{2i\theta}$

$re^{-\lambda\theta} = k$

∴ $r = ke^{\lambda\theta}$, logarith. spiral



is W-curve, part. case [illegible] Klein [illegible]

W-curve [illegible] Encyk. III 3, Heft 2/3

Loria, Spezielle Kurven II, p. 178 [illegible]

63. Extended transformation, differential invariants

$\xi(xy)$, $\eta(xy)$ [illegible] xy [illegible] analyt. funct. [illegible]

$\delta x = \xi\delta t,\quad \delta y = \eta\delta t$

$Uf = \xi p + \eta q$

[illegible] infinitesimal point transformation

[illegible] $y' = \frac{dy}{dx},\ y'' = \frac{dy'}{dx}$ [illegible] vary

[illegible]

$\delta y' = \delta\frac{dy}{dx} = \frac{\delta dy\cdot dx - dy\cdot\delta dx}{(dx)^2}$

$= \frac{d\delta y\cdot dx - dy\cdot d\delta x}{(dx)^2} = \frac{d\delta y}{dx} - y'\frac{d\delta x}{dx}$

$= \left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right)\delta t$ $\qquad \eta_x = \frac{\partial\eta}{\partial x}$

$\delta y' = [\eta_x + \eta_y y' - y'(\xi_x + \xi_y y')]\,\delta t$

∴ $\delta y' = [\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2]\,\delta t$

Def. $\delta x = \xi\delta t,\ \delta y = \eta\delta t,\ \delta y' = [\quad]\,\delta t$ [illegible]

1st extended transform [illegible] $\delta y''$ [illegible] 2nd ext. [illegible]

$$\delta y'' = \frac{d\,\delta y'}{dx} - y''\frac{d\,\delta x}{dx}$$

$$\delta y'' = [\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'']\,\delta t$$

$\delta y'' =$ ––

<u>Theorem</u> [illegible]
[illegible] inf. [illegible] inf. proj. [illegible]

Proof. $y'' = 0$ [illegible]
$\therefore y''$ [illegible] $y = ax + b$ [illegible]
$y'' = 0$ [illegible] condit. [illegible] $\delta y'' \equiv 0$ [illegible]

$\delta y'' = [\qquad\qquad]\,\delta t$

coeff. [illegible] $\equiv 0$.

$$\begin{cases} \eta_{xx} = 0, & \eta_{yy} - 2\xi_{xy} = 0 \\ \xi_{yy} = 0, & \xi_{xx} - 2\eta_{xy} = 0 \end{cases}$$

$\xi = Xy + X_0$, X [illegible] funct.
$\eta = Yx + Y_0$ (1)

[illegible] $\eta_y - 2\xi_x = X_1$,
$\xi_x - 2\eta_y = Y_1$

[illegible] solve [illegible]
$$\begin{cases} 3\xi_x = -2X_1 - Y_1 \\ 3\eta_y = -2Y_1 - X_1 \end{cases}$$

(1) [illegible]

$$\begin{cases} 3X'y + 3X_0' = -2X_1 - Y_1 \\ 3Y_1'x + 3Y_0' = -2Y_1 - X_1 \end{cases}$$ c, d, const.

[illegible] Y_1 [illegible] y [illegible] linear f. $Y_1 = -3cy - 3d$ [illegible]
[illegible] $X_1 = -3ax - 3b$
[illegible] y coeff [illegible]
$X' = c$, $Y' = a$.
[illegible] $X = cx + \gamma$, $Y = ay + \alpha$,
[illegible] $X_0' = 2ax + 2b + d$
$Y_0' = 2cy + 2d + b$
$\therefore X_0 = ax^2 + (2b + d)x + \beta$,
$Y_0 = cy^2 + (2d + b)y + \delta$,
[illegible] $\xi = Xy + X_0$, $\eta = Yx + Y_0$ [illegible]
$\xi = (cx + \gamma)y + ax^2 + (2b + d)x + \beta$.
$\eta = (ay + \alpha)x + cy^2 + (2d + b)y + \delta$.

constant [illegible] notat. [illegible]

$$Uf = (a_1 + c_1x + d_1y + h_1x^2 + k_1xy)p + (b_1 + e_1x + g_1y + h_1xy + k_1y^2)q$$

[illegible] inf. proj. transf. [illegible]

Remark. [illegible] $y'' = 0$ [illegible] inf. proj. [illegible] $y'' = 0$ [illegible] y'' [illegible] <u>differential (proj.) invariant</u> [illegible]

Remark. Conic [illegible] collineat. [illegible] conic [illegible] tan Conic diff eq. [illegible] invariant [illegible]

Conic: $y = mx + n + \sqrt{Ax^2+2Bx+C}$ [illegible]

$$y'' = \frac{AC-B}{(Ax^2+2Bx+C)^{\frac{3}{2}}}$$

$$\therefore (y'')^{-\frac{2}{3}} = (AC-B)^{-\frac{2}{3}}(Ax^2+2Bx+C)$$

$$\therefore \frac{d^3}{dx^3}\left[(y'')^{-\frac{2}{3}}\right] = 0$$

$$40y'''^3 - 45y''y'''y'''' + 9y''^2y^{v} = 0 \quad \text{Halphen}$$

[illegible] Conic diff inv. [illegible] Alphen [illegible]

[illegible] proj. diff inv. [illegible] diff. proj. [illegible]

[illegible]

64. Affine group

Theorem 1 affine transf. analyt. express.

$$\begin{cases} x_1 = a_1x + b_1y + c_1 \\ y_1 = a_2x + b_2y + c_2 \end{cases} \quad \Delta \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

[illegible] aff. tr. [illegible] l. at ∞ [illegible]

coll. [illegible] parallel line [illegible]

$$lx_1 + my_1 = k \ \text{(arbitr. const.)}$$

[illegible] parallel l. [illegible]

$$l'x + m'y = k'$$

l', m' [illegible] k [illegible] k [illegible] variable

[illegible] coll. [illegible]

$$l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = k(a_3x + b_3y + c_3)$$

$$(la_1 + ma_2 - ka_3)x + (lb_1 + mb_2 - kb_3)y = kc_3 - lc_1 - mc_2$$

x, y coeff. k [illegible]

$$a_3 = 0, \quad b_3 = 0$$

$$\therefore x_1 = a_1x + b_1y + c_1$$

$$y_1 = a_2x + b_2y + c_2$$

[illegible] express. [illegible] aff. tr. [illegible]

Theorem 2 aff. tr. [illegible] 6-termed cont. group [illegible] group [illegible] general coll. group [illegible] subgroup [illegible]

$$a_1 = 1, \quad b_1 = 0 \quad c_1 = 0$$

$$a_2 = 0, \quad b_2 = 1, \quad c_2 = 0$$

$$a_1 = 1 + \alpha_1\delta t \quad b_1 = \beta_1\delta t \quad c_1 = \gamma_1\delta t$$

$$a_2 = \alpha_2\delta t \quad b_2 = 1 + \beta_2\delta t \quad c_2 = \gamma_2\delta t$$

$$\delta x = (\alpha_1x + \beta_1y + \gamma_1)\delta t, \quad \delta y = (\alpha_2x + \beta_2y + \gamma_2)\delta t$$

$$Uf \equiv (\alpha_1x + \beta_1y + \gamma_1)p + (\alpha_2x + \beta_2y + \gamma_2)q$$

[illegible] infinit. aff. tr. [illegible] 6 termed [illegible] general case [illegible] const. [illegible] vanish [illegible] subgr. [illegible]

Theorem 3 aff. tr. [illegible] $5y'''^2 - 3y''y'''' = 0$ [illegible] inv. [illegible]

[illegible] inf. tr. [illegible]

∴ parabola [illegible] aff. tr. [illegible] parabola [illegible]

[illegible] parabola [illegible] eq. $y = mx + n + \sqrt{2Bx + C}$ [illegible]

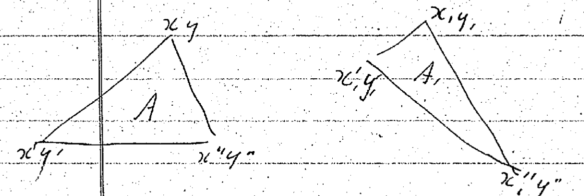
$$y'' = \frac{-B}{(2Bx+C)^{\frac{3}{2}}} \qquad (y'')^{-\frac{2}{3}} = (-B^2)^{-\frac{2}{3}}(2Bx+C)$$

$$\frac{d^2}{dx^2}\left[(y'')^{-\frac{2}{3}}\right] = 0 \quad \text{or} \quad 5y'''^2 - 3y''y^{iv} = 0$$

aff. [illegible] parabola [illegible] parabola [illegible] aff. [illegible] aff. [illegible]

<u>Theorem 4</u> closed curve [illegible] area [illegible] aff. [illegible] relative invariant [illegible]

[illegible] Δ [illegible] A_0 [illegible] area [illegible] A_1 [illegible]



$$A_1 = \Delta \cdot A \qquad \Delta \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\begin{cases} x_1 = a_1 x + b_1 y + c_1 \\ y_1 = a_2 x + b_2 y + c_2 \end{cases} \qquad \begin{cases} x_1' = a_1 x' + b_1 y' + c_1 \\ y_1' = a_2 x' + b_2 y' + c_2 \end{cases}$$

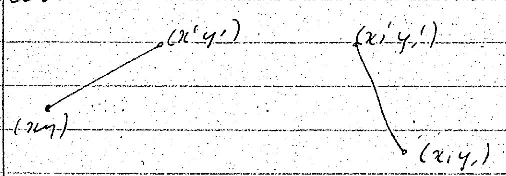
$$\begin{cases} x_1'' = a_1 x'' + b_1 y'' + c_1 \\ y_1'' = a_2 x'' + b_2 y'' + c_2 \end{cases}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_1' & x_1'' \\ y_1 & y_1' & y_1'' \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix}$$

$$2A_1 = \Delta \cdot 2A$$

65. <u>Group of motion.</u> (rigid body)

[illegible] pt (x, y) [illegible] (x', y') [illegible] distance [illegible] absolute invar. [illegible] coll. [illegible]



infinite dist. [illegible]

[illegible] coll. [illegible] pt at ∞ [illegible] pt at ∞ [illegible] affine [illegible]

$$\begin{cases} x_1 = a_1 x + b_1 y + c_1 \\ y_1 = a_2 x + b_2 y + c_2 \end{cases}$$

$$\begin{cases} x_1' = a_1 x' + b_1 y_1' + c_1 \\ y_1' = a_2 x' + b_2 y' + c_2 \end{cases}$$

dist. [illegible] absolute value [illegible]

$$(x_1 - x_1')^2 + (y_1 - y_1')^2 = [a_1(x - x') + b_1(y - y')]^2 + [a_2(x - x') + b_2(y - y')]^2$$

$$= (x - x')^2 + (y - y')^2 \text{ [illegible]}$$

$a_1^2 + a_2^2 = 1$, $b_1^2 + b_2^2 = 1$, $a_1 b_1 + a_2 b_2 = 0$

[illegible] necessary + suff. cond. [illegible] orthogonal transf.

[illegible] $\begin{cases} a_1 = \cos\alpha \\ a_2 = \sin\alpha \end{cases}$ $\begin{cases} b_1 = \cos\beta \\ b_2 = \sin\beta \end{cases}$ [illegible]

[illegible] $\cos(\alpha - \beta) = 0$ $\therefore \beta = \alpha + (n+1)\frac{\pi}{2}$

$n = 0, \pm 1, \pm 2, \ldots$

$\begin{cases} b_1 = (-1)^{n+1} \sin\alpha \\ b_2 = (-1)^{n+1} \cos\alpha \end{cases}$

[illegible] case [illegible]

1° n, even. $a_1 = \cos\alpha$, $a_2 = \sin\alpha$, $b_1 = -\sin\alpha$, $b_2 = \cos\alpha$

$\begin{cases} x_1 = x\cos\alpha - y\sin\alpha + a, \\ y_1 = x\sin\alpha + y\cos\alpha + b \end{cases}$ congruence [illegible]

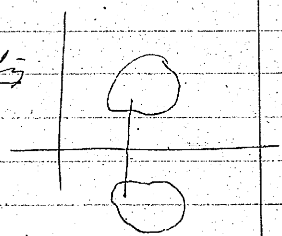
[illegible] axis [illegible] displac. [illegible] rotation [illegible]

ii° n, odd. $\begin{cases} x_1 = x\cos\alpha + y\sin\alpha + a \\ y_1 = x\sin\alpha - y\cos\alpha + b \end{cases}$

[illegible] y [illegible] $-y$ [illegible] replace [illegible]

x axis [illegible] reflection [illegible] symmetry

[illegible]



1. II 1.3 [illegible] group [illegible]

Comb. [illegible] dist. [illegible] const [illegible]

[illegible] continuous group [illegible]

Def. Congruence [illegible] 1. case [illegible] cont. group [illegible] the 3-termed group [illegible] Group of motion [illegible] 1. II 1.3 [illegible] group [illegible] Mixed (gemischte) Group [illegible]

group of motion [illegible] identical [illegible] $a = 0$, $b = 0$, $\alpha = 2n\pi$

$\alpha = 2n\pi + \lambda\delta t$, $a = \mu\delta t$, $b = \nu\delta t$ [illegible] variation [illegible]

$\delta x = x_1 - x = x\cos(2n\pi + \lambda\delta t) - y\sin(2n\pi + \lambda\delta t)$

$+ \mu\delta t - x$

$= x[1 - \frac{\lambda^2\delta t^2}{2!} + \ldots) - 1] - y(\frac{\lambda\delta t}{1!} \ldots)] + \mu\delta t$

2nd diff. [illegible]

$\delta x = (-\lambda y + \mu)\delta t$

[illegible] $\delta y = (\lambda x + \nu)\delta t$

infin. motion [illegible] symbol

$Uf \equiv (-\lambda y + \mu)p + (\lambda x + \nu)q$

$\equiv \lambda(xq - yp) + \mu p + \nu q$

[illegible] inf. motion [illegible] $xq - yp$, p, q [illegible] compose [illegible]

$xq - yp$ [illegible] origin [illegible] rotation [illegible] (§ 61 Ex. 2)

p, q [illegible] translation [illegible]

$\mu p + \nu q$ [illegible] finite form [illegible] $x_1 = x + \mu t$

$y_1 = y + \nu t$

$\frac{dx_1}{\mu} = \frac{dy_1}{\nu} = dt$,

$x_1 = x$, $y_1 = y$ for $t = 0$ [illegible]

... infinit. mot. ... Euclidean space ... space ... Helmholtz ... Raum-problem ... Lie (Transformationsgruppe III) ... Helmholtz ... non-euclidean space ... Lobatsch. Riemann space ... Hilbert, Poincaré ... Klein ... Höhere Geom. II ...

66. Differential invariants of motion

I. §.63 ... $Uf = \xi p + \eta q$ ...

$$\delta y' = [\eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2] \delta t$$

... mot. ...

$$\xi = -\lambda y + \mu, \quad \eta = \lambda x + \nu$$

$$\delta y' = \lambda (1 + y'^2) \delta t$$

1st extended mot. symbol

$$\lambda \left(x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x}\right) + \mu \frac{\partial f}{\partial x} + \nu \frac{\partial f}{\partial y} + \lambda (1 + y'^2) \frac{\partial f}{\partial y'}$$

... 1st order dif. eq.

$\varphi(x, y, y') = 0$ ...

$\varphi(x y y') \equiv y' - \omega(x, y) = 0$ ...

... eq. ... mot. ... invariant ... $\delta\varphi = 0$ ...

$$\lambda \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x}\right) + \mu \frac{\partial \varphi}{\partial x} + \nu \frac{\partial \varphi}{\partial y} + \lambda (1 + y'^2) \frac{\partial \varphi}{\partial y'} = 0 \text{ for } \varphi = 0$$

... λ, μ, ν ...

$$\frac{\partial \varphi}{\partial x} = 0, \quad \frac{\partial \varphi}{\partial y} = 0, \quad x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} + (1 + y'^2) \frac{\partial \varphi}{\partial y'} = 0$$

$$\frac{\partial \varphi}{\partial y'} = 1, \quad \therefore 1 + y'^2 = 0$$

$$y' = \pm i$$

$$x \pm i y = \text{Const.}$$

... minimal line ... $y' = \pm i$ ... (1st ord. dif. eq. ... mot. ... inv.) ... singularity ...

Theorem 1. Circular pt. at ∞ ... mot. ... min. l. ... l. at ∞ ... inv. ...

Theorem 2. Circular pt. at ∞ ... inv. ... collineat. ... mot. ... similar transf. ... circ. pt. at ∞ ... inv. ... coll. ... aff. tr. ...

... aff. tr. ... $x + iy = \text{Const}$ ... $x_1 + iy_1 = \text{Const}$, $x - iy = \text{Const}$ ... $x_1 - iy_1 = \text{Const}$ ...

$$x_1 = \rho (x \cos\alpha - y \sin\alpha + a)$$

$$y_1 = \rho (x \sin\alpha + y \cos\alpha + b) \quad \rho \text{ Const.}$$

... mot. ... similar transf. ... mot. ... ρ ... similar tr. ...

Remark. x axis ... symmetry ... invari. ... circular pt. ...

[illegible] circ. pts [illegible] [illegible]

Def. mot^n, sym., similar [illegible] group [illegible] circ. pts at ∞ [illegible] inv. [illegible] group general form: $x_1 = \rho(x\cos\alpha \mp y\sin\alpha + a)$, $y_1 = \rho(x\sin\alpha \pm y\cos\alpha + b)$ [illegible] group [illegible] subgroup [illegible] 4 termed mixed group [illegible] Klein [illegible] *Hauptgruppe* [illegible] (Principal gr.)

II. $\delta y'' = 3\lambda y' y'' \delta t$ (mot^n [illegible])

2nd extended mot^n [illegible] symbol

$$\lambda\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) + \mu\frac{\partial f}{\partial x} + \nu\frac{\partial f}{\partial y} + \lambda(1+y'^2)\frac{\partial f}{\partial y'} + 3\lambda y'y''\frac{\partial f}{\partial y''}$$

[illegible] 2nd order [illegible] dif. eq.

$\varphi \equiv y'' - \omega(x, y, y') = 0$ [illegible]

[illegible] ~~mot^n~~ [illegible] mot^n [illegible] inv. [illegible] λ, μ, ν [illegible] $)_{\lambda=\varphi} = 0$ for $\varphi = 0$

$$\frac{\partial\varphi}{\partial x} = 0,\ \frac{\partial\varphi}{\partial y} = 0,\ \left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) + (1+y'^2)\frac{\partial\varphi}{\partial y'} + 3y'y''\frac{\partial\varphi}{\partial y''} = 0$$

φ [illegible] x, y [illegible] $\varphi \equiv y'' - \omega(y')$ ~~= 0~~ [illegible]

[illegible] $-(1+y'^2)\dfrac{d\omega}{dy'} + 3y'\omega = 0$

[illegible] $\omega = \text{Const.}(1+y'^2)^{\frac{3}{2}}$

[illegible] $\dfrac{y''}{(1+y'^2)^{\frac{3}{2}}} = \text{Const.}$

2nd ord. diff. eq. [illegible] mot^n [illegible] inv. [illegible]

[illegible] curvature [illegible]

<u>Theorem 3.</u> Plane [illegible] mot^n [illegible] 2nd ord. dif. inv. [illegible] curvature [illegible]

[illegible] 3rd ord. dif. inv. [illegible]

$\Omega\left(\frac{1}{r}, \frac{dr}{ds}\right) = 0$

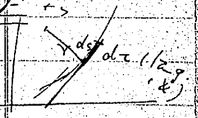
$\frac{1}{r}$ curvature, s curve length (lang., arc length) [illegible]

Ω arbitrary funct. ds [illegible] $d\tau$ [illegible]

[illegible] curve [illegible] <u>intrinsic (natural) equat^n</u> [illegible]

[illegible] curve [illegible] eq. [illegible]

[illegible] curve [illegible] mot^n [illegible] eq.



[illegible] nat. eq. [illegible] curve [illegible]

plane curve, space curve, surf. natural eq. [illegible] Scheffers, Diff. Geom. I p. 42, p. 201; II p. 302, Lie, Cont. Gr. p. 666–716

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space [illegible] all [illegible] analogous [illegible]

67 Infinitesimal proj. [illegible] of the line

x x_1

[illegible] line [illegible] x [illegible] x_1 [illegible] proj. [illegible]

$x_1 = \dfrac{ax+b}{cx+d}$ $ad - bc \neq 0$

$a = 1, b = 0, c = 0, d = 1.$ identical el.

$a = 1 + \alpha_1\delta t,\ b = \beta_1\delta t,\ c = \gamma_1\delta t,\ d = 1 + \delta_1 dt$

$\delta x = (\alpha + \beta x + \gamma x^2)\,\delta t$

$Uf \equiv (\alpha + \beta x + \gamma x^2)\frac{\partial f}{\partial x}$

3 termed cont. group [illegible]

line [illegible] any one pt. any two pts. any 3 pts [illegible] proper sense [illegible] 3 pts [illegible] 4 pts [illegible] proper [illegible]

x, x', x'', x''' [illegible]

$\Omega(x_1, x_1', x_1'', x_1''') = \Omega(x, x', x'', x''')$

[illegible] condit. [illegible]

$$\frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial x'}dx' + \frac{\partial\Omega}{\partial x''}dx'' + \frac{\partial\Omega}{\partial x'''}dx''' = 0$$

$$(\alpha+\beta x+\gamma x^2)\frac{\partial\Omega}{\partial x} + (\alpha+\beta x'+\gamma x'^2)\frac{\partial\Omega}{\partial x'} + (\cdots)\frac{\partial\Omega}{\partial x''} + (\;)\frac{\partial\Omega}{\partial x'''} = 0$$

$$\alpha\left(\frac{\partial\Omega}{\partial x} + \frac{\partial\Omega}{\partial x'} + \frac{\partial\Omega}{\partial x''} + \frac{\partial\Omega}{\partial x'''}\right)$$
$$+\beta\left(x\frac{\partial\Omega}{\partial x} + x'\frac{\partial\Omega}{\partial x'} + x''\frac{\partial\Omega}{\partial x''} + x'''\frac{\partial\Omega}{\partial x'''}\right)$$
$$+\gamma\left(x^2\frac{\partial\Omega}{\partial x} + x'^2\frac{\partial\Omega}{\partial x'} + x''^2\frac{\partial\Omega}{\partial x''} + x'''^2\frac{\partial\Omega}{\partial x'''}\right) = 0$$

α, β, γ arbitrary const. [illegible] coeff. [illegible] 0 [illegible]

$\frac{\partial\Omega}{\partial x} + \cdots = 0$ (1)

$x\frac{\partial\Omega}{\partial x} + \cdots = 0$ (2)

$x^2\frac{\partial\Omega}{\partial x} + \cdots = 0$ (3)

[illegible] diff. eq. [illegible] solution [illegible]

(1) [illegible] Ω [illegible] Lagrange's linear partial eq. [illegible]

$dx = dx' = dx'' = dx'''$

$u' \equiv x' - x,\; u'' \equiv x'' - x,\; u''' \equiv x''' - x$ const.

Ω [illegible] $\Omega(u', u'', u''')$ [illegible]

[illegible] Ω [illegible] (2) [illegible] satisfy [illegible]

$$u'\frac{\partial\Omega}{\partial u'} + u''\frac{\partial\Omega}{\partial u''} + u'''\frac{\partial\Omega}{\partial u'''} = 0$$

$$\frac{\partial u'}{u'} = \frac{\partial u''}{u''} = \frac{\partial u'''}{u'''}$$

$\log u' = \log u'' + \log c$

$v \equiv \frac{u''}{u'} = \text{const.}$

$w \equiv \frac{u'''}{u'} = \text{const.}$

$\therefore\; \Omega$ [illegible] $\Omega(v, w)$ [illegible]

[illegible] (3) [illegible]

[illegible] $(v-1)v\frac{\partial\Omega}{\partial v} + (w-1)w\frac{\partial\Omega}{\partial w} = 0$

$$\frac{dv}{(v-1)v} = \frac{dw}{(w-1)w}$$

$$\frac{v-1}{v} : \frac{w-1}{w} = \text{const.}$$

$\Omega = \Phi\left(\frac{v-1}{v} : \frac{w-1}{w}\right)$ Φ, arbitrary f.

$$\therefore\; \Omega = \Phi\left(\frac{x''-x'}{x''-x} : \frac{x'''-x'}{x'''-x}\right)$$

[illegible] part. dif. eq. [illegible] solution [illegible]

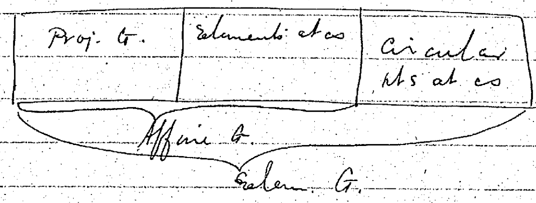
4 given pts [illegible] anharmonic ratio [illegible] arb. f. [illegible]

Theorem [illegible] general proj. [illegible]
4 pts [illegible] inv. [illegible] double ratio [illegible] arbitrary [illegible]
Proj. [illegible] double ratio [illegible]

6.8. Klein's principle for classification of Geometry

Klein [illegible] 1872 Vergleichende Betrachtungen über neuere geom. Forschungen (Erlangen [illegible]) [illegible] cont. group [illegible] mixed group [illegible] Hauptgroup [illegible] invariant [illegible] figure [illegible] Geometry of group [illegible] inv. [illegible] collineation, correlation [illegible] mixed group [illegible] inv. thing [illegible] project. geom. [illegible] affine [illegible] cont. group [illegible] invariant thing [illegible] affine geom. [illegible] motion, symmetry, similar [illegible] mixed group (Hauptgruppe) [illegible] inv. thing [illegible] äquiforme Geometrie [illegible] elementargeom. [illegible]

Proj. G. { aff. G. { Elem. g.



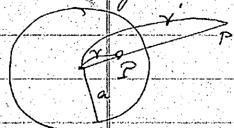
[illegible] Inzidenz [illegible] proj. G. parallel [illegible] Heffter [illegible] Parallelmetrik [illegible] Orthogonalmetrik [illegible] Inzid. + Parallelmetrik [illegible] Affine G. [illegible] Inz. + Par. + Orthog. [illegible] Elem. G. [illegible] (circular invol. [illegible] ⊥ el. [illegible] minimal line, circular pts at ∞ [illegible] define [illegible]) [illegible] Heffter [illegible] Cayley [illegible] A sixth memoir on quantics (algebraic form) (1859) [illegible] Projective geom. is all geometry [illegible] Klein [illegible] 2nd order curve [illegible] thing [illegible] proj. g. [illegible] real, proper [illegible] conic [illegible] tangent, pole + polar, conjugate pts or lines [illegible] aff. g. [illegible] Ellipse, Hyp. Parabola [illegible] centric, diameter, asymptote [illegible] elem. g. [illegible] circle, rectangular hyperb. [illegible] axis, focus [illegible] geom. [illegible] systematic [illegible] so [illegible] Heffter-Kohler, 1st, 2nd rank [illegible]

Lehrb. d. analyt. G. Bd. I (1905) [illegible]

Remark 1. Hauptgruppe [illegible] group of moti[on] [illegible] Geom. of motion elem. g. [illegible] group standpt in geom. of moti[on] [illegible] Study, Ü. Bewegungsinvarianten u. elementare Geometrie, Leipz. Ber. 1896 p. 649

Study, Geom. d. Dynamen (1901) Abs. II [illegible]

Remark 2. Klein [illegible] group [illegible] [illegible] gen. [illegible] Inversion [illegible]



$rr' = a^2$ Hauptgruppe circle, sphere, adjoin [illegible] inversion [illegible] group [illegible] Conformal group [illegible] (Inver[illegible]) [illegible] Conformal geom. [illegible] (or geom. of reciprocal radii, niedere Kugelgeom.) (höhere Kugelgeom. [illegible] circle, sphere [illegible] Orientat[illegible])

[illegible]

Lie, Cont. Gr.

" , Diff. gleichg

Klein, Höhere Geom. II

Klein, Classif. principle [illegible]

Klein, Höh. G. I.

" , Elem. math. II,

Encykl. III, Heft 2, G. Fano, Kontin. geom. Gruppen. Die Gruppentheorie als geom. Einteilungsprinzip.

Chapter XV. Correlation & Null systems

69. Principle of duality in the plane

(Correl. [illegible] contact transf. & null syst. [illegible])

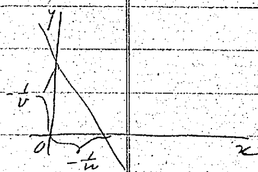
u, v [illegible] parameter [illegible] plane [illegible] line [illegible]

(1) $ux + vy + 1 = 0$

u, v [illegible] line [illegible] coeff. u, v [illegible] line [illegible] line coordinates [illegible] (Plücker 1830)

u, v geom. meaning [illegible]

$$\frac{x}{-\frac{1}{u}} + \frac{y}{-\frac{1}{v}} = 1$$



x, y [illegible] intercept [illegible] reciprocal [illegible] opposite [illegible]

Plücker [illegible] line coord. [illegible] plane element [illegible] line [illegible] Cartesian geom. [illegible] pt. [illegible] element [illegible]

[illegible]

	pt plane	line plane
current coord.	x, y	u, v
linear eq.	line	point
linear form + [illegible]	flat pencil	point range

collineat [illegible] line coord [illegible] transform [illegible]

(2) $x_1 = \frac{a_1x+b_1y+c_1}{a_3x+b_3y+c_3}, \quad y_1 = \frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}$

[illegible] (1) [illegible] (3) [illegible]

(1) $ux+vy+1=0$ (u,v) (3) $u_1x_1+v_1y_1+1=0$ (u_1,v_1)

(2) [illegible] x, y [illegible] solve [illegible]

$x = \frac{A_1x_1+A_2y_1+A_3}{C_1x_1+C_2y_1+C_3}, \quad y = \frac{B_1x_1+B_2y_1+B_3}{C_1x_1+C_2y_1+C_3}$

$A_1, B_1, \ldots$ [illegible] $\delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, a_1, b_1 [illegible] algebraic complements [illegible]

[illegible] x, y [illegible] (1) [illegible]

$u(A_1x_1+A_2y_1+A_3)+v(B_1x_1+B_2y_1+B_3)+(C_1x_1+C_2y_1+C_3)$ [illegible]

$\therefore \frac{A_1u+B_1v+C_1}{A_3u+B_3v+C_3}x_1 + \frac{A_2u+B_2v+C_2}{A_3u+B_3v+C_3}y_1+1=0$

(3) [illegible]

(4) $u_1 = \frac{A_1u+B_1v+C_1}{A_3u+B_3v+C_3}, \quad v_1 = \frac{A_2u+B_2v+C_2}{A_3u+B_3v+C_3}$

Theorem 1. plane [illegible] flat pencil [illegible] flat pencil [illegible] line transf [illegible] (line [illegible] line [illegible])

$u_1 = \frac{\alpha_1u+\beta_1v+\gamma_1}{\alpha_3u+\beta_3v+\gamma_3}, \quad v_1 = \frac{\alpha_2u+\beta_2v+\gamma_2}{\alpha_3u+\beta_3v+\gamma_3}$ [illegible]

[illegible] §60 Theorem 1 [illegible] collineat [illegible]

x, y [illegible] u, v [illegible] x_1, y_1 [illegible] u_1, v_1 [illegible]

[illegible] Theorem [illegible]

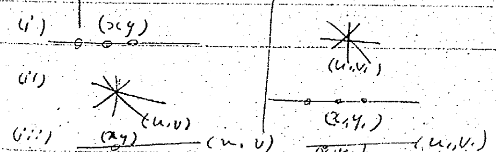
Theorem 2. (i) [illegible] pt range [illegible] all pts x, y [illegible] flat pencil line u_1, v_1 [illegible] (ii) [illegible] flat pencil [illegible] all lines u, v [illegible] pt range pts x_1, y_1 [illegible]

(iii) line u, v [illegible] pt (x, y) [illegible] pt (x_1, y_1) [illegible] line (u_1, v_1) [illegible] transf [illegible] expression [illegible]

(1) $u_1 = \frac{a_1x+b_1y+c_1}{a_3x+b_3y+c_3}, \quad v_1 = \frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}$;

(2) $x_1 = \frac{A_1u+B_1v+C_1}{A_3u+B_3v+C_3}, \quad y_1 = \frac{A_2u+B_2v+C_2}{A_3u+B_3v+C_3}$

$\Delta \equiv \begin{vmatrix} a_1 & & \\ & b_2 & \\ & & c_3 \end{vmatrix} \neq 0$, A_1 [illegible] Δ [illegible] a_1 [illegible] algebraic complement.



Proof. (i) [illegible] assumption [illegible] x, y [illegible] linear eq. [illegible]
u, v [illegible] linear eq. [illegible] §60
there [illegible] x_1, y_1 [illegible] u_1, v_1 [illegible] (1)
[illegible] (ii) [illegible] (2) $x_1 = \dfrac{\alpha_1 u + \beta_1 v + \gamma_1}{\alpha_3 u + \beta_3 v + \gamma_3}$, $y_1 = \dfrac{\alpha_2 u + \beta_2 v + \gamma_2}{\alpha_3 u + \beta_3 v + \gamma_3}$

[illegible] (ii) [illegible] (3) $ux + vy + 1 = 0$ [illegible] (4) $u_1 x + v_1 y + 1 = 0$
[illegible] (3) [illegible] (1) [illegible] $x = \dfrac{A_1 u + B_1 v + A_3}{C_1 u + C_2 v + C_3}$,

$y = \dfrac{B_1 x + B_2 v + B_3}{C_1 x + C_2 v + C_3}$ [illegible]

$\dfrac{A_1 u + B_1 v + C_1}{A_3 u + B_3 v + C_3} u_1 + \dfrac{A_2 u + B_2 v + C_2}{A_3 u + B_3 v + C_3} v_1 + 1 = 0$ [illegible]

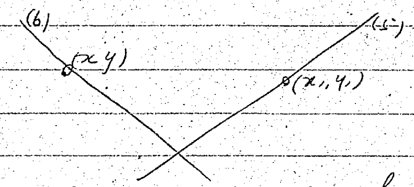
[illegible] u_1, v_1 [illegible] linear eq. [illegible] fixed pt [illegible]
flat pencil [illegible] pt [illegible] x_1, y_1 [illegible]
(4) [illegible] x_1, y_1 [illegible]

$x_1 = \dfrac{A_1 u + B_1 v + C_1}{A_3 u + B_3 v + C_3}$, $y_1 = \dfrac{A_2 u + B_2 v + C_2}{A_3 u + B_3 v + C_3}$

[illegible] (2) [illegible]
$A_1 = \alpha_1$, $B_1 = \beta_1$, ---
[illegible]

Def. [illegible] theorem [illegible] pt [illegible] line [illegible] 3 [illegible]
rel. [illegible] Möbius [illegible] *Allgemeine Dualität* [illegible]
(1827) [illegible] Correl. [illegible]

(1) [illegible] (4) [illegible]
(5) $(a_1 x + b_1 y + c_1) x_1 + (a_2 x + b_2 y + c_2) y_1 + (a_3 x + b_3 y + c_3) = 0$
[illegible] (x, y) [illegible] (x_1, y_1) [illegible] bilinear rel. [illegible]
allg. Dualität [illegible] *Äquatio directrix* [illegible]
(Plücker) (*generating eq.* [illegible])
pt (x, y) [illegible] fix [illegible] eq. [illegible] (x_1, y_1) [illegible] current
coord. [illegible] line [illegible]
(6) $(a_1 x_1 + a_2 y_1 + a_3) x + (b_1 x_1 + b_2 y_1 + b_3) + (c_1 x_1 + c_2 y_1 + c_3) = 0$
[illegible] (x_1, y_1) [illegible] fix [illegible] (x, y) [illegible] current [illegible]
[illegible] line [illegible]



[illegible] eq. [illegible] correl. [illegible] eq. [illegible]
[illegible]

70. *Particular Cases.*

gen. eq. [illegible] x, y [illegible] x_1, y_1 [illegible] symmetrical
[illegible] transform [illegible] express [illegible]
[illegible] $(x, y) \xrightarrow{T} (x_1, y_1)$
[illegible] duality [illegible] *involutory*, *involutional* $(x_1, y_1) \xleftarrow{T} (x, y)$
[illegible] $T^2 = 1$

involutory duality の condition は次の如し

I (5) (6) より

$a_2 = b_1,\ a_3 = c_1,\ b_3 = c_2$

即ち $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ が symmetric となる場合

II. $a_1 = b_2 = c_3 = 0,\ a_2 = -b_1,\ a_3 = -c_1,\ b_3 = -c_2$ となる場合

II の場合 (6) は

$(-b_1 y_1 - c_1)x + (b_1 x_1 - c_2)y + (c_1 x_1 + c_2 y_1)$

は identically に vanish する故 general eq. ... ... となり ... ...

故に involutory duality は I に限る

I の場合 ... generating eq.

$a_1 x x_1 + b_2 y y_1 + a_2(y x_1 + x y_1) + a_3(x + x_1) + c_2(y + y_1) + c_3 = 0$

となる (X, Y current coord.)

$a_1 X^2 + 2a_2 XY + b_2 Y^2 + 2a_3 X + 2c_2 Y + c_3 = 0$

なる conic に関する x_1, y_1 の polar なり

Theorem: reciprocal polar は algebraic duality の involutory なる particular case なり。algebraic duality の involutory なるは reciprocal polar なり (Möbius, general duality; Poncelet, reciprocal polar)

例 $x^2 + y^2 + 1$ なる circle に関する $xx_1 + yy_1 - 1 = 0$ を generating eq. とする Poncelet's circle に関する reciprocal polar. thus ... imaginary circle $x^2 + y^2 + 1 = 0$ に関する ... gener. eq

$xx_1 + yy_1 + 1 = 0$ となる。homogeneous coord. で

表せば $x^2 + y^2 + z^2 = 0$, $xx_1 + yy_1 + zz_1 = 0$ と ...

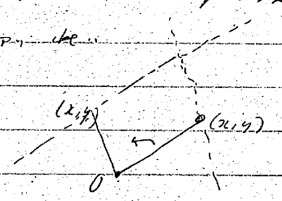
Chasles: ... $2y - x^2 = 0$ なる parabola に関して

$xx_1 - y - y_1 = 0$ (Salmon, § 322)

algebraic D. ... ...

$xy_1 - yx_1 = k$ ...

$\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = k$



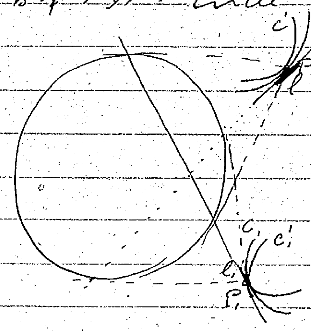
此 ... area const

(x, y) は fix ...

(x_1, y_1) ...

71. Contact transformations

... circle $x^2 + y^2 - 1 = 0$ に関する reciprocal



... curve C, C' が

P に於て touch する

reciprocal curve C_1, C_1'

C_1, C_1' は P ... tangent

... P_1 に於て touch する

(lineal, linear)

C, C' [illegible] P [illegible] tangent [illegible] line element l [illegible] common [illegible]

[illegible] C, C' [illegible] line el. l_1 [illegible] line el. l [illegible] pt P [illegible] l_1 [illegible] pole [illegible] line el. [illegible] point P_1 [illegible] l_1 pole [illegible] l. l_1 [illegible] rel. [illegible] polar reciprocal [illegible] line el. [illegible] transformat. [illegible]

line el. [illegible] pt (x, y) [illegible] direct. $y' (= \frac{dy}{dx})$ [illegible] rough [illegible] line el. coord. [illegible] x, y, y' [illegible] point [illegible] line [illegible] plane element [illegible] line el. [illegible] plane el. [illegible] line el. [illegible] transform [illegible] Lie (1870) [illegible]

[illegible] $x^2 + y^2 - 1 = 0$ [illegible]

(1) $xx_1 + yy_1 - 1 = 0$ [illegible] gen. eq. [illegible] line el. [illegible] (x, y, y'), (x_1, y_1, y_1') [illegible]

(1) [illegible] $y_1 = \frac{1}{y}(-xx_1 + 1) = -\frac{x}{y} x_1 + \frac{1}{y}$
(direct.)

(2) $y_1' = -\frac{x}{y}$

[illegible] $y = -\frac{x_1}{y_1} x + \frac{1}{y_1}$

(3) $y' = -\frac{x_1}{y_1}$

(1), (2), (3) [illegible]

$$x_1 = \frac{-y'}{y - xy'}, \quad y_1 = \frac{1}{y - xy'}, \quad y_1' = -\frac{x}{y}$$ [illegible]

[illegible] l [illegible] l_1 [illegible] transf. [illegible] polar reciprocat. [illegible] line el. [illegible] transf. [illegible]

P_1 (x_1, y_1, y_1') — $(x_1 + dx_1, y_1 + dy_1, y_1' + dy_1')$

$$dx_1 = \frac{-y\,dy' + y'dy - y'^2 dx}{(y - xy')^2} \qquad dy_1 = \frac{-dy + y'dx + x\,dy'}{(y - xy')^2}$$

$$\therefore \; dy_1 - y_1' dx_1 = \frac{(-y + xy')dy + (y - xy')y'dx}{y(y - xy')^2}$$

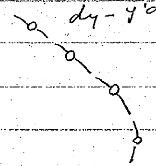
$$\therefore \; dy_1 - y_1' dx_1 = \rho(dy - y'dx)$$

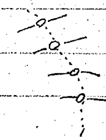
where $\rho = \frac{-1}{y(y - xy')}$, diff. [illegible]

Def. [illegible] line el. x, y, y' [illegible] $dy - y'dx = 0$ [illegible] line el. [illegible] Elementverein (Union of el.) [illegible] (x, y [illegible] pt. [illegible] curve [illegible] line el. [illegible])

Theor. 1 Elementverein [illegible] curve [illegible] l. el. [illegible]

$dy - y'dx = 0$ $dy - y'dx \neq 0$ [illegible]

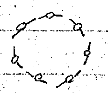




Ex. $x^2+y^2=1$, $x+yy'=0$

$x\,dx+y\,dy=0$

$$\frac{dy}{dx}=-\frac{x}{y}=y'$$

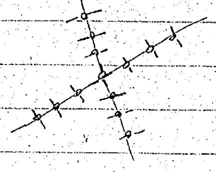


x $\quad y+xy'=0$

$x^2-y^2=0$ [illegible]

$$\frac{dy}{dx}=\frac{x}{y} \qquad y'=-\frac{y}{x}$$

$\therefore \dfrac{dy}{dx}\neq y'$ Elementare [illegible]



Def. line el. transf.

$x_1=X(x\,y\,y')$, $y_1=Y(x\,y\,y')$, $y_1'=P(x\,y\,y')$

[illegible] Elementare $dy-y'dx=0$ [illegible] corresp.

El. ver. $dy_1-y_1'dx_1=0$ [illegible] transf. [illegible]

Contact transformation [illegible] touch [illegible]

[illegible] curve [illegible] touch [illegible] curve [illegible]

(touch [illegible] l. el. [illegible])

contact transf. [illegible] condition [illegible]

$$dy_1-y_1'dx_1=dY-P\,dX$$
$$=(Y_x\,dx+Y_y\,dy+Y_{y'}\,dy')-P(X_x\,dx+X_y\,dy+X_{y'}\,dy')=(Y_x-PX_x)dx+(Y_y-PX_y)dy$$
$$+(Y_{y'}-PX_{y'})dy'$$
$$=0 \text{ for } dy-y'dx=0 \text{ [illegible]}$$

= [illegible] dx, dy, dy' [illegible] linear homog. [illegible]

$dy-y'dx=0$ $\quad dy-y'dx+0\cdot dy'$ [illegible] linear homog. [illegible] 0 [illegible] 0 [illegible]

[illegible] const. [illegible]

$$dy_1-y_1'dx_1=\rho(x\,y\,y')\,(dy-y'dx)$$

[illegible] contact transf. [illegible] 0 [illegible] 0 [illegible]

Theorem 2 linear el. transf. [illegible] contact tr. [illegible] necess. + suff. cond. $dy_1-y_1'dx_1=\rho(dy-y'dx)$ [illegible] (Lie, 1872)

Theorem 3 allg. Dualit. [illegible] contact transf. [illegible]

[illegible] gen. eq. $(a_1x+b_1y+c_1)x_1+(a_2x+b_2y+c_2)y_1+(a_3x+b_3y+c_3)=0$ [illegible] line el. [illegible]

$$x_1=\frac{A_1y'-B_1+C_1(y-xy')}{A_3y'-B_3+C_3(y-xy')}$$

$$y_1=\frac{A_2y'-B_2+C_2(y-xy')}{A_3y'-B_3+C_3(y-xy')}$$

$$y_1'=-\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

[illegible] condition [illegible] contact transf. [illegible]

Ex. $2y - x^2 = 0$ [illegible] polar recipr.

$y + y_1 - x x_1 = 0$ gen eq.

$x_1 = y'$, $y_1 = xy' - y$, $y' = x$

[illegible] diff eq. [illegible]

Legendre [illegible] transf.

Clairaut's eq. [illegible]

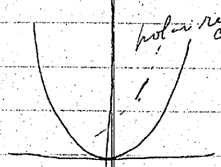
$y = xy' + f(y')$

[illegible] parabola [illegible]

polar recip. [illegible]

$y_1 + f(x_1) = 0$

[illegible] polar recip. [illegible] solution



Lie [illegible] contact tr. [illegible] Plücker [illegible] gen eq. [illegible]

Geom. d. Berührungstransf. I [illegible]

72. Principle of duality in space

space [illegible] duality [illegible] plane [illegible] duality [illegible]

[illegible] (x, y, z) pt coord. (u, v, w) plane coord.

$ux + vy + wz + 1 = 0$

analy. [illegible] expression

$u_1 = \frac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4}$, $v_1 = \frac{a_2x \dots}{a_4x \dots}$, $w_1 = \frac{a_3x \dots}{a_4x \dots}$

$x_1 = \frac{A_1u + B_1v + C_1w + D_1}{A_4u + B_4v + C_4w + D_4}$, $y_1 = \dots$, $z_1 = \dots$

gen eq.

$(a_1x + b_1y + c_1z + d_1)x_1 + (a_2x + \dots + d_2)y_1 + (a_3x + \dots + d_3)z_1 + (a_4x + b_4y + c_4z + d_4) = 0$ (Möbius 1833)

[illegible] duality [illegible] involutory [illegible] condition

I

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$ [illegible] symmetric [illegible]

[illegible] 2nd order surface [illegible] polar reciprocal [illegible]

II [illegible] determinant [illegible] skew-sym [illegible]

$$\begin{vmatrix} 0 & -C & B & -D \\ C & 0 & -A & E \\ -B & A & 0 & G \\ -D & -E & -G & 0 \end{vmatrix}$$

[illegible] gen eq. [illegible]

$(Bz - Cy + D)x_1 + (Cx - Az + E)y_1 + (Ay - Bx + G)z_1 - (Dx + Ey + Gz) = 0$

[illegible] $x_1 = x$, $y_1 = y$, $z_1 = z$ [illegible] eq. $\frac{0}{0}$

[illegible] xyz [illegible] plane [illegible]

pt [illegible] corr. plane [illegible] plane [illegible] corr.

pt [illegible] Null-syst.

Def. [illegible] duality [illegible] Null system [illegible]
(Möbius 1837, Lehrbuch der Statik [illegible])

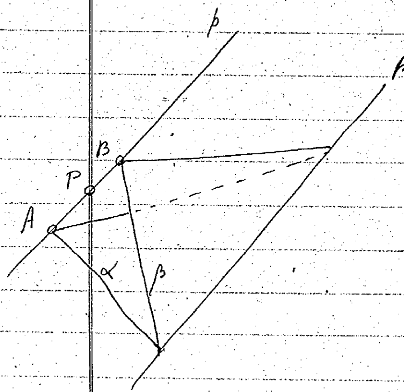
Nullsyst. [illegible] §37 [illegible] Ex. 2 [illegible]

[illegible]

Theorem. space [illegible] invol. correl. ($T^2 = 1$) [illegible] polar reciprocation [illegible] Null syst [illegible] (synth. [illegible] Reye II [illegible])

73. Fundamental properties of the nullsystem

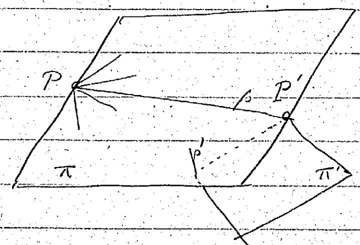
Null syst. const. A B C D E F G [illegible] space [illegible] ∞^5 [illegible] p [illegible] line [illegible] A, B [illegible] A, B [illegible] corr. pl [illegible] α, β [illegible]



p_1 [illegible] p_1 [illegible] p [illegible] P [illegible] p [illegible] corr. plane π [illegible] p_1 [illegible] axial pencil [illegible]

Def. [illegible] P p, p_1 [illegible] Null syst. [illegible] Conjugate [illegible] Poinsot (Sur la composition des moments, etc., 1806) [illegible]

[illegible] pl P [illegible] corr. pl π



π [illegible] P [illegible] vertex [illegible] flat pencil [illegible] line p [illegible] p_1 [illegible] P′ [illegible]

P′ [illegible] corr. pl. π′ [illegible] p [illegible] line [illegible] p_1 [illegible] (conjugate) p_1 [illegible] π, π′ [illegible] p, P, P′, [illegible] π [illegible] P [illegible] π′ [illegible] P′ [illegible] p [illegible] P′ [illegible]

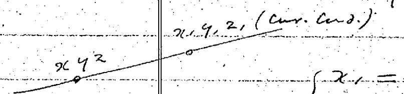
[illegible] P′ [illegible] P [illegible] p [illegible] P [illegible] P, p [illegible] P P′ [illegible] p [illegible] p [illegible] self-conjugate line [illegible]

Theorem 1. [illegible] Nullsyst [illegible] plane [illegible] corr. pt [illegible] line [illegible] self-conj. [illegible]

Def. [illegible] plane [illegible] Null plane, [illegible] pt [illegible] Null point [illegible] line [illegible] Null line [illegible]

Null line [illegible] eq. [illegible]

$\begin{cases} x = rz + \rho \\ y = sz + \sigma \end{cases}$ … co^4 … line … Null syst. …



$\begin{cases} x_1 = rz_1 + \rho \\ y_1 = sz_1 + \sigma \end{cases}$

$$\frac{x_1 - x}{r} = \frac{y_1 - y}{s} = \frac{z_1 - z}{1}$$

xyz Null pl. …

$(Bz - Cy + D)x_1 + (Cx - \ldots \qquad = 0$

… Null l. … Null pl.

$r(Bz - Cy + D) + s(Cx - Az + E) + (Ay - Bx + G) = 0$

$A(y - sz) - B(x - rz) + C(sx - ry) + Dr + Es + G = 0$

$A\sigma - B\rho + C(s\rho - r\sigma) + Dr + Es + G = 0$

Theorem … line $x = rz + \rho$, $y = sz + \sigma$ … null l. … condition …

Remark … space co^4 line … Null syst. … line … co^3 …

Def. … curve … tangent … Null line … Null syst. curve …

… eq …

curve any pt (xyz) consecutive pt $(x_1 y_1 z_1)$

$x_1 = x + dx \quad y_1 = y + dy \quad z_1 = z + dz$

(xyz) … $(x_1 y_1 z_1)$ … line … tangent …

… Null line … condition …

$(Bz - Cy + D)x_1 + (Cx - Az + E)y_1 + (Ay - Bx + G)z_1 - (Dx + Ey + Gz) = 0$

$(Bz - Cy + D)x + (\quad)y + (\quad)z - (\quad) = 0$

identical …

$(Bz - Cy + D)(x - x_1) + (\quad)(y - y_1) + (\quad)(z - z_1) = 0$

$(Bz - Cy + D)dx + (Cx - Az + E)dy + (Ay - Bx + G)dz = 0$

… Null syst. … curve … dif eq.

… P, Q, R … xyz … f …

$Pdx + Qdy + Rdz = 0$ … Pfaff's eq …

Null syst. … curve … particular case …

74 Infinitesimal motion in space

Kinematics … Null syst. …

Null syst. … eq. … simplify …

space xyz … inf. mot. …

… dist … transf. …

$\delta x = \xi(xyz)\,dt, \quad \delta y = \eta(xyz)\,dt,$

$\delta z = \zeta(xyz)\,dt$ … variat. …

… arc element $ds^2 = dx^2 + dy^2 + dz^2$ … invari…

11. inf. rot. in space [illegible]

$\delta(dx^2 + dy^2 + dz^2) = 0$

i.e. $dx\,\delta dx + dy\,\delta dy + dz\,\delta dz = 0$

$dx\, d\delta x + dy\, d\delta y + dz\, d\delta z = 0$

$dx\, d\xi + dy\, d\eta + dz\, d\zeta = 0$

$(d\xi = \xi_x dx + \xi_y dy + \xi_z dz$

$d\eta = \cdots$

$d\zeta = \cdots$

[illegible]

$\xi_x dx^2 + \eta_y dy^2 + \zeta_z dz^2 + (\xi_y + \eta_x)\, dx\, dy + (\xi_x + \zeta_z)\, dx\, dz$
$+ (\eta_z + \zeta_y)\, dy\, dz = 0$

dx, dy, dz arbitrary [illegible]

[illegible] 0 [illegible]

(1) $\xi_x = 0$ (2) $\eta_y = 0$, (3) $\zeta_z = 0$,

(4) $\xi_y + \eta_x = 0$ (5) $\xi_x + \zeta_z = 0$, (6) $\eta_z + \zeta_x = 0$

(1) [illegible] ξ [illegible] x = independent $\xi(y, z)$

(2) [illegible] $\eta(x, z)$ (3) [illegible] $\zeta(x, y)$

(4) [illegible] y [illegible] diff [illegible] $\xi_{yy} + \eta_{xy} = 0$

η [illegible] xz [illegible] $\eta_{xy} = 0$ $\xi_{yy} = 0$

(5) [illegible] z [illegible] part. dif [illegible] $\xi_{xz} + \xi_{zz} = 0$

ζ [illegible] xy [illegible] $\xi_{zz} = 0$

ξ [illegible] y, z [illegible] y [illegible] z [illegible] dif [illegible] $= 0$ [illegible]

[illegible] linear [illegible]

$\xi = b_1 y + c_1 z + d_1$

$\eta = a_2 x + c_2 z + d_2$

$\zeta = a_3 x + b_3 y + d_3$

cond. (4)(5)(6) [illegible] dif [illegible]

$b_1 + a_2 = 0$

$a_3 + c_1 = 0$

$c_2 + b_3 = 0$

$\xi = Bz - Cy + D$

$\eta = Cx - Az + E$

$\zeta = Ay - Bx + G$ [illegible]

[illegible]

$\delta x = (Bz - Cy + D)\, dt$

$\delta y = (Cx - Az + E)\, dt$

$\delta z = (Ay - Bx + G)\, dt$

[illegible] inf. rot. [illegible]

$Bz - Cy + D = \rho A$

$Cx - Az + E = \rho B$

$Ay - Bx + G = \rho C$ [illegible]

[illegible] A, B, C [illegible]

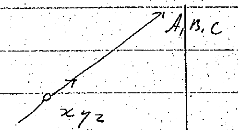
$AD + BE + CG = \rho(A^2 + B^2 + C^2)$

[illegible] ρ [illegible]

[illegible]

[illegible] space [illegible] line [illegible]

$$\frac{x+E-\rho B}{A} = \frac{y-D+\rho A}{B} = \frac{z}{C}$$



[illegible] space [illegible]

$$\frac{\delta x}{A} = \frac{\delta y}{B} = \frac{\delta z}{C} = \rho\,\delta t$$

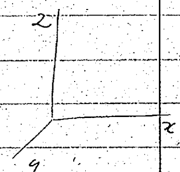
[illegible] xyz [illegible] xyz [illegible] ABC [illegible]

[illegible] inf. mot. [illegible]

[illegible] Axis of inf. mot. [illegible]

[illegible] axis [illegible] Z-axis [illegible]

Z [illegible] eq. $x=0,\ y=0$



[illegible] $\delta x = 0,\ \delta y = 0$

[illegible] $\delta x =$, $\delta y =$, $\delta z =$ [illegible]

$B=0,\ D=0,\ A=0,\ E=0$

$\therefore\ \delta x = -y\,\delta t,\quad \delta y = x\,\delta t,\quad \delta z = k\,\delta t$

$C\,\delta t \to \delta t,\quad G\,\delta t \to k\,\delta t$ [illegible]

$\delta x = -y\,\delta t,\ \delta y = x\,\delta t$ [illegible] origin [illegible] rotation [illegible], $\delta z = k\,\delta t$ translation [illegible]

[illegible] spiral mot., screw [illegible] (z axis)

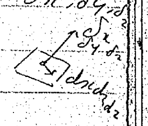
Theorem 1. inf. m. [illegible] screw mot. [illegible]

([illegible]) Mozzi, 1763

non-Euclidean sp. [illegible] $ds^2 = dx^2 + dy^2 + dz^2$ [illegible] arc el. [illegible]

R [illegible] differential form [illegible] Helmholtz [illegible] problem [illegible] (rigid body mot. [illegible] space [illegible])

[illegible] (xyz) [illegible] inf. m. [illegible] direct. $\delta x, \delta y, \delta z$ [illegible] dx, dy, dz [illegible] normal condit.



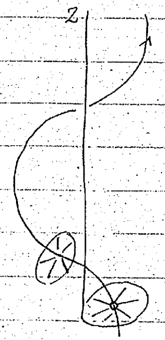
$$dx\,\delta x + dy\,\delta y + dz\,\delta z = 0$$

[illegible] $\delta x\ \delta y\ \delta z$ [illegible]

$$(Bz - Cy + D)\,dx + (Cx - Az + E)\,dy + (Ay - Bx + G)\,dz = 0$$

[illegible] Null l. [illegible] cun [illegible]

Theorem 2. Null l. [illegible] inf. mot. [illegible] normal [illegible]



[illegible] inf. m. [illegible] Null syst. [illegible] Chasles (1837) [illegible] null syst. [illegible] normal form [illegible]

$$-y\,dx + x\,dy + k\,dz = 0$$

$(x\,dy - y\,dx + k\,dz = 0$ [illegible]$)$

k [illegible] value [illegible] null system [illegible] chief parameter [illegible]
axis [illegible] chief par. [illegible] screw [illegible] null system [illegible]

Ex. 1. helix

$x = a\cos t,\ y = a\sin t,\ z = \frac{a^2}{k}t + \text{const.}$ [illegible]

$dx = -a\sin t\,dt,\ dy = a\cos t\,dt,\ dz = -\frac{a^2}{k}dt$

$a^2(\cos^2 t + \sin^2 t)\,dt - k\frac{a^2}{k^2}\,dt = 0$

Null system [illegible] curve [illegible] tangent [illegible] null line [illegible]

Ex. 2. Space cubic

$x = t,\ y = \frac{1}{2}t^2,\ z = \frac{1}{6k}t^3 + \text{const.}$

Chapter XVI Line Geometry

75. Line Coordinates

[illegible] line [illegible] space [illegible] ∞^4 [illegible] (1) $x = rz + \rho,\ y = sz + \sigma$ [illegible] line space [illegible] study [illegible] Plücker [illegible] ruled surface, null line [illegible] point coord. x, y, z [illegible] space plane space [illegible] u, v, w [illegible] line space [illegible] line coord. [illegible] r, s, ρ, σ [illegible] line unique [illegible] r, s, ρ, σ [illegible]

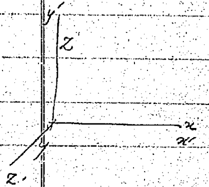
[illegible] r, s, ρ, σ [illegible] line coord. [illegible] (Plücker [illegible] homogeneous coord. [illegible] gen. [illegible] Optics [illegible] line gen. [illegible])

[illegible] r, s, ρ, σ [illegible] n-th degree [illegible] algebraic eq.

$\phi(r, s, \rho, \sigma) = 0$ [illegible]

$x' = x,\quad y' = z,\quad z' = y$

[illegible] y, z [illegible]



$x' = ry' + \rho,\ z' = sy' + \sigma$

$x' = \frac{r}{s}z' + \frac{s\rho - r\sigma}{s},\ y' = \frac{1}{s}z' - \frac{\sigma}{s}$

$x' = r'z' + \rho',\ y' = s'z' + \sigma'$ [illegible]

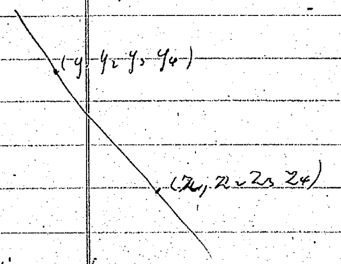
$r' = \frac{r}{s},\ s' = \frac{1}{s},\ \rho' = \frac{s\rho - r\sigma}{s},\ \sigma' = -\frac{\sigma}{s}$ [illegible]

$\phi(r', s', \rho', \sigma') = 0$

[illegible] n-th degree [illegible] homog. [illegible] degree [illegible] Plücker (1865) $r, s, \rho, \sigma, r\sigma - s\rho (= \eta)$ [illegible] quantities [illegible] line coordinates [illegible] coord. [illegible] n-th degree [illegible] eq. [illegible] collineat [illegible] degree [illegible]

[illegible] homogeneous line coord. [illegible]

[illegible] space [illegible] homog. coord. [illegible] $(y_1, y_2, y_3, y_4),\ (z_1, z_2, z_3, z_4)$ [illegible] current coord ; $(x_1\ x_2\ x_3\ x_4)$ [illegible]



$$p_{ik} = \begin{vmatrix} y_i & y_k \\ z_i & z_k \end{vmatrix} \quad (i,k = 1, 2, 3, 4)$$

[illegible]

$$p_{ii} = 0, \quad p_{ik} = -p_{ki}$$

i, k [illegible]

$p_{12},\ p_{13},\ p_{14},\ p_{23},\ p_{24},\ p_{34}$

[illegible] 6 quantities [illegible] point y, z [illegible] line

homogeneous line coordinates [illegible]

[illegible] 6 [illegible] independent [illegible]

$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$ [illegible]

[illegible] Coord. [illegible] Grassmann 1844 [illegible]

[illegible] plane [illegible] line homog.

coord [illegible]

[illegible] plane [illegible] $u_1\ u_2\ u_3\ u_4$ [illegible] current coord [illegible]

$v_1\ v_2\ v_3\ v_4$, $w_1\ w_2\ w_3\ w_4$ [illegible] plane [illegible]

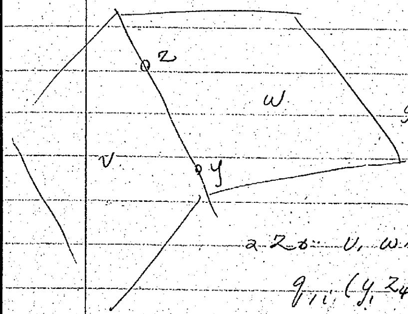
$$q_{ik} = \begin{vmatrix} v_i & v_k \\ w_i & w_k \end{vmatrix} \qquad q_{ii} = 0, \quad q_{ik} = -q_{ki}$$

$q_{12},\ q_{13},\ q_{14},\ q_{23},\ q_{24},\ q_{34}$

[illegible] plane [illegible] line homog. coord [illegible]

[illegible] Plücker 1865 [illegible]

[illegible] p [illegible] q [illegible]



y [illegible] v [illegible] const. [illegible]

$$v_1y_1 + v_2y_2 + v_3y_3 + v_4y_4 = 0 \quad \Big| w_i$$

$$w_1y_1 + w_2y_2 + w_3y_3 + w_4y_4 = 0 \quad \Big| v_i$$

$$(v_iw_i - w_iv_i)y_1 + \cdots = 0$$

$$\begin{array}{l|l} z_k & q_{1i}y_1 + q_{2i}y_2 + q_{3i}y_3 + q_{4i}y_4 = 0 \\ -y_k & q_{1i}z_1 + q_{2i}z_2 + q_{3i}z_3 + q_{4i}z_4 = 0 \end{array}$$

[illegible] v, w [illegible]

$$q_{1i}(y_1z_k - z_1y_k) + \cdots = 0$$

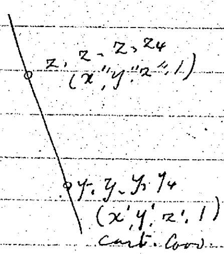
$$q_{1i}p_{1k} + q_{2i}p_{2k} + q_{3i}p_{3k} + q_{4i}p_{4k} = 0$$

$i, k = 1, 2, 3, 4$ [illegible] linear eq. [illegible] solution

$$\frac{q_{12}}{p_{34}} = \frac{q_{13}}{p_{42}} = \frac{q_{14}}{p_{23}} = \frac{q_{23}}{p_{14}} = \frac{q_{42}}{p_{13}} = \frac{q_{34}}{p_{12}}$$

[illegible] p [illegible] q [illegible]

homog. line coord. [illegible] non-homog. l. coord. [illegible]



$$p_{12} = x'y'' - y'x''$$

$$p_{23} = y'z'' - z'y''$$

$$p_{31} = z'x'' - x'z''$$

$$p_{41} = x'' - x'$$

$$p_{42} = y'' - y'$$

$$p_{43} = z'' - z'$$

$x = rz + \rho,\quad y = sz + \sigma$ ...

$x' = rz' + \rho,\quad y' = sz' + \sigma$

$x'' = rz'' + \rho,\quad y'' = sz'' + \sigma$

$$r = \frac{x'-x''}{z'-z''} = \frac{p_{14}}{p_{43}},\quad s = \frac{y'-y''}{z'-z''} = \frac{p_{24}}{p_{34}}$$

$$\rho = \frac{z'x''-x'z''}{z'-z''} = \frac{p_{31}}{p_{34}},\quad \sigma = -\frac{y'z''-z'y''}{z'-z''} = \frac{p_{32}}{p_{34}}$$

$$\eta = s\rho - r\sigma = \frac{p_{24}p_{31} - p_{14}p_{32}}{p_{34}^2} = \frac{p_{12}}{p_{34}}$$

non-homog. coord ... p_{34} ...

26 Proj. transformation

space ... coll. homog. point coord. (x_1, x_2, x_3, x_4) ...

$$x_i' = a_{i1}x_1 + \dots + a_{i4}x_4 \quad (i = 1, \dots, 4)$$

... coll. ... y

$$y_i' = a_{i1}y_1 + \dots + a_{i4}y_4$$

$$z_i' = a_{i1}z_1 + \dots + a_{i4}z_4$$

$$p_{ik}' = \begin{vmatrix} y_i' & y_k' \\ z_i' & z_k' \end{vmatrix} = \sum_{\mu,\nu}^{1\dots4} a_{i\mu}a_{k\nu} \begin{vmatrix} y_\mu & y_\nu \\ z_\mu & z_\nu \end{vmatrix}$$

$$\therefore\ p_{ik}' = \sum_{\mu,\nu}^{1\dots4} a_{i\mu}a_{k\nu}p_{\mu\nu}$$

p ... linear transform ...

$$r' = \frac{p_{14}'}{p_{34}'},\quad s' = \frac{p_{24}'}{p_{34}'},\ \dots$$

$$r' = \frac{\sum a_{1\mu}a_{4\nu}p_{\mu\nu}}{\sum a_{3\mu}a_{4\nu}p_{\mu\nu}} = \frac{\text{lin. f. of } (r, s, \rho, \sigma, \eta)}{\text{lin. f. of } (r, s, \rho, \sigma, \eta)}$$

$s' = \dots$

...

hence

Theorem Alg. eq. $\phi(p_{ik}) = 0$ ... $\phi(r, s, \rho, \sigma, \eta) = 0$ ... coll. ... degree ...

Plücker ... line ...

Remark plane ... (x_1, x_2, x_3) pt. homog. coord ... $x_3 = 0$ ... line at ∞ ... invariant ... coll. affine transf. ... $x_1^2 + x_2^2 = 0$ ... $x_3 = 0$ ... coll. ... gen. ... dimension ... homog. coord 3 ...

proj. line geom. ... p_{ik} ... 3 homog. coord ... 5 dim. space ... figure $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$ ... invariant ... proj. geom. ... Klein ...

77. Line complex, congruence, ruled surface

pt space $(x_1\, x_2\, x_3\, x_4)$ [illegible] pt [illegible] co^3 [illegible]

cond. 3 [illegible] F=0 [illegible] rel. [illegible] co^2 [illegible] surface [illegible]

$F_1=0,\ F_2=0$ [illegible] co^1 [illegible] curve [illegible]

[illegible] dual [illegible] plane space $(u_1\, u_2\, u_3\, u_4)$ [illegible]

co^3, plane [illegible]

F=0 [illegible] co^2 plane [illegible] surface [illegible] envelope

$F_1=0\ F_2=0$ co^1 [illegible] developable ~~surface~~ plane surface

[illegible]

[illegible] correspond [illegible] line space [illegible]

co^4 p_{ik} line space

F=0 co^3 line complex

$F_1=0, F_2=0$ co^2 [illegible] line complex [illegible] line congruence (Linien-kongruenz od. Strahlensystem)

$F_1=0\ F_2=0\ F_3=0$ co^1 [illegible] ruled surface [illegible]

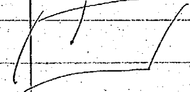
[illegible] surface [illegible] line fig. [illegible]

[illegible] F=0 [illegible] alg. eq. [illegible] deg. [illegible]

complex [illegible] deg [illegible] linear comp. quadratic complex [illegible]

[illegible] Null line co^3 [illegible] Line complex [illegible]

2



surface [illegible] only one line [illegible]

[illegible] co^2 [illegible]

line congruence [illegible] surface [illegible]

normal [illegible]

Optics [illegible] source [illegible] Ray [illegible] surface [illegible]

[illegible] Ray [illegible] line congruence [illegible]

Line congruence theory [illegible] optics [illegible] Malus (1808)

Hamilton (1830) [illegible] Bruns [illegible] geometrical

optics [illegible] line congruence theory [illegible]

78. Linear complex

Theorem 1. Linear complex [illegible] null line [illegible]

Line complex [illegible] Line complex [illegible]

null l. [illegible] eq.

$$A\sigma - B\rho + C(s\rho - r\sigma) + Dr + Es + G = 0$$

line coord. $A\frac{p_{12}}{p_{34}} - B\frac{p_{31}}{p_{34}} + C\frac{p_{21}}{p_{14}} + D\frac{p_{14}}{p_{34}} + E\frac{p_{24}}{p_{34}} + G = 0$

$$\therefore\ a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{24}p_{24} + a_{34}p_{34} = 0$$

[illegible] Linear [illegible]

[illegible] linear comp. [illegible] most general form [illegible]

Theorem 2. [illegible] l. p_{ik}, p'_{ik} [illegible] nec. + suf.

Cond. [illegible]

$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{23}p'_{14} + p_{42}p'_{13} + p_{34}p'_{12} = 0$

$p \cdot p' = y\,z\ p'\ y'z'$ [illegible]

$p \cdot p'$ [illegible] on plane [illegible]

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \\ z'_1 & z'_2 & z'_3 & z'_4 \end{vmatrix} = 0$$

[illegible] Laplace [illegible] develop [illegible]

Theorem 3 — linear complex $(ap) = \sum_{ik}^{1 \ldots 4} a_{ik}p_{ik} = 0$ [illegible] a certain line [illegible]

$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 0$

linear complex [illegible] p_{ik} [illegible]

$p_{12}p'_{34} + \cdots = 0$

$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$

$p_{34} = K\,a_{12}, \ldots$

$p'_{12}p_{34} + p'_{13}p_{42} + p'_{14}p_{23} = 0$

$a_{34}a_{12} + a_{13}a_{42} + a_{14}a_{23} = 0$

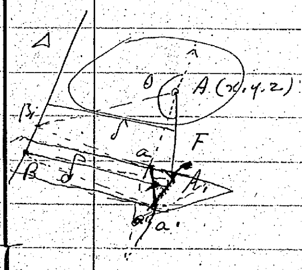
[illegible] (a_{ik}) [illegible] line coord. [illegible]

Def. [illegible] linear comp. [illegible] special complex [illegible]

Theorem 4. $A \equiv a_{12}a_{34} + a_{14}a_{42} + a_{14}a_{23} \neq 0$ [illegible] invariant [illegible]

Def. A [illegible] linear complex [illegible] Invariant [illegible] special complex [illegible]

Ex. Statics [illegible] applicat.



A [illegible] resultant force act.

F [illegible] component X, Y, Z

Δ [illegible] any line [illegible] space [illegible]

Δ [illegible] force [illegible] moment

moment $= F\delta \sin\theta$

δ [illegible] Δ [illegible] F [illegible] shortest distance

θ [illegible] F [illegible] Δ [illegible] angle

δ [illegible] AA_1 [illegible] Δ parallel plane [illegible]

Δ [illegible] B [illegible] $\perp$ [illegible] δ [illegible]

B [illegible] Δ normal plane [illegible] normal plane [illegible]

AA' [illegible] orthogonal proj. [illegible] aa' [illegible] aa' [illegible] B [illegible] $\perp$ [illegible] δ [illegible]

Δ [illegible] B_1 [illegible] vol $(AA_1BB_1) =$ vol (aa_1BB_1)

$= \frac{1}{3}\left(\frac{1}{2}\,aa_1\,\delta\right)\cdot BB_1$

$= \frac{1}{6}\,\delta \cdot BB_1 \cdot AA'_1 \sin\theta$

Moment $= \dfrac{6\ \mathrm{vol}(AA_1BB_1)}{BB_1}$

$A(x\,y\,z),\ A_1(x+X,\ y+Y,\ z+Z)\ B(x'y'z'),\ B_1(x''y''z'')$

$L = yZ - zY,\ M = zX - xZ,\ N = xY - yX$ [illegible]

rigid body force [illegible] X, Y, Z [illegible] L, M, N [illegible]

$$\pm 6\ \text{vol}\ (AA_1 BB_1) = \pm \begin{vmatrix} x & y & z & 1 \\ x+X & y+Y & z+Z & 1 \\ x' & y' & z' & 1 \\ x'' & y'' & z'' & 1 \end{vmatrix}$$

$$= \pm [(x''-x')L + (y''-y')M + (z''-z')N + (y'z''-y''z')X + (z'x''-x'z'')Y + (x'y''-x''y')Z]$$

$$\text{Moment} = \pm \frac{L p_{41} + M p_{42} + N p_{43} + X p_{23} + Y p_{13} + Z p_{12}}{\sqrt{p_{41}^2 + p_{42}^2 + p_{43}^2}}$$

[illegible]

Theorem (Möbius 1837) [illegible] force [illegible] moment [illegible] 0 [illegible] line [illegible] Null syst. [illegible]

Null syst. Null: moment [illegible] 0 [illegible]

79. Linear congruence

[illegible] line complex $(ap) = 0$ $(bp) = 0$ [illegible] ∞^2 [illegible] line [illegible] totality [illegible] linear congruence [illegible] linear eq. [illegible]

$$(ap) + \lambda(bp) = 0$$

λ, arbitrary const. [illegible]

[illegible] complex [illegible] special complex [illegible] Inv. $= 0$

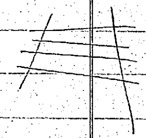
$$A + M\lambda + B\lambda^2 = 0$$

$A = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23},\ B = b_{12}b_{34} + \cdots$

$M = a_{12}b_{34} + \cdots$

[illegible] quadr. eq. root λ_1, λ_2 [illegible]

$$\begin{cases} (ap) + \lambda_1(bp) = 0 \\ (ap) + \lambda_2(bp) = 0 \end{cases}$$



[illegible] linear congr. [illegible] special complex [illegible] congruence [illegible]

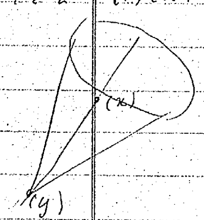
Theorem [illegible] line congr. [illegible] fixed line [illegible]

Cor. [illegible] lin. comp. $(ap) = 0$ $(bp) = 0$ $(cp) = 0$ [illegible] 3 [illegible] line [illegible] 2nd ord. [illegible] ruled surface [illegible] syst. [illegible]

80. Quadratic complex

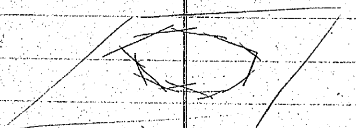
$\Phi_2(p_{ik}) = 0$ [illegible] quadratic homog. eq. [illegible] complex [illegible] plane geom. [illegible] 2nd degree curve [illegible] space geom. [illegible] quadr. surface [illegible] line geom. [illegible] quadr. complex [illegible] systematic stud. [illegible] Plücker (1868–9) (Neue Geom.) Klein (1868, 70) (Klein [illegible] Line coordinate [illegible] imaginary unit [illegible]) [illegible]

3 space [illegible] any pt $(y_1\, y_2\, y_3\, y_4)$ [illegible] complx [illegible]
pt [illegible] $(x_1\, x_2\, x_3\, x_4)$ [illegible]



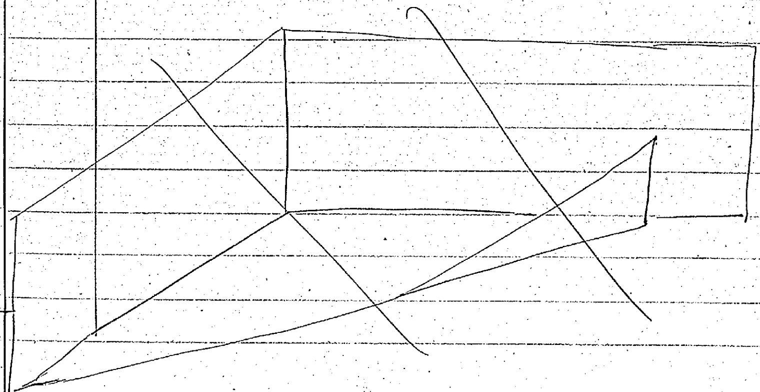
$\phi_2(x_i y_k - y_i x_k) = 0$
[illegible] eq. [illegible] $x_1\, x_2\, x_3\, x_4$ [illegible] 2nd dg.
homog. eq. [illegible] 2nd dg.
surface [illegible] (y) [illegible]
2nd dg. cone [illegible]
[illegible] dual [illegible] one plane (v) [illegible]
coord. $(v_1\, v_2\, v_3\, v_4)$ [illegible] plane [illegible] complx line
coord. [illegible] $u_1\, u_2\, u_3\, u_4$ [illegible]

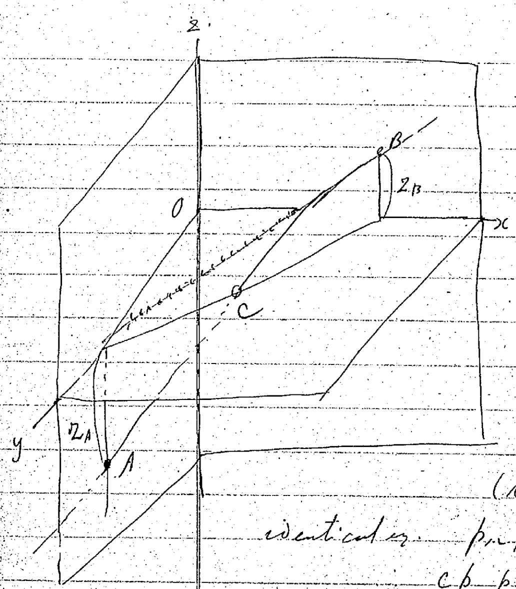
$\psi_2(q_i\, q_k) = 0$ $\quad \psi_2(u_i v_k - u_k v_i) = 0$ [illegible]
[illegible] $u_1\, u_2\, u_3\, u_4$ [illegible] 2nd dg. homog. eq. [illegible]
line [illegible] one plane [illegible] line
2nd class cone [illegible] envelope [illegible]



Theorem 1. Quadr. complx [illegible] one pt [illegible] line [illegible]
2nd ~~ord~~ class cone [illegible] generate [illegible] one plane [illegible]
line [illegible] 2nd ~~dg.~~ class cone [illegible] envelope [illegible]
Remark [illegible] theorem, proof
n th ~~dg.~~ class complx [illegible] apply [illegible]

flat pencil or one point [illegible] (Nullsystem)
[illegible] quadr complx [illegible] 2nd dg. cone [illegible]
[illegible] pt [illegible] plane [illegible] break up [illegible] pt [illegible] singular
pt [illegible] quadr complx [illegible] line [illegible] sing. pt
[illegible] sing. pt. locus [illegible] ~~conic~~ quartic surface
[illegible] Kummer's surf. [illegible] (1864) [illegible]
surf. [illegible] 16 [illegible] node (double pts) [illegible]
quartic surface [illegible] (Klein [illegible])
Quadr. complx [illegible]
particular case [illegible]
[illegible] line aggregate [illegible] tetrahedron [illegible]
[illegible] pt [illegible] double ratio [illegible] constant given
value [illegible] line [illegible] aggr. [illegible] eq.
[illegible] $x = 0,\ y = 0,\ z = 0$, plane
at ∞ [illegible]





$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{AC}{BC}$$

li. eq.

$$x = rz + \rho, \quad y = sz + \sigma$$

$$\frac{AC}{BC} = \frac{z_A}{z_B} = \frac{-\frac{\rho}{r}}{-\frac{\sigma}{s}} = \frac{s\rho}{r\sigma}$$

$$\therefore \frac{s\rho}{r\sigma} = \text{const} = \frac{a-c}{b-c}$$

$a \neq b \neq c$ arbitrary const.

$$(b-c)p_{24}p_{31} + (a-c)p_{14}p_{23} = 0$$

identically $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$

$$cp_{12}p_{34} = -cp_{24}p_{31} - cp_{14}p_{23}$$

$$a\,p_{34}p_{12} + b\,p_{24}p_{31} + c\,p_{12}p_{14} = 0 \quad (a \neq b \neq c)$$

<u>Def.</u> ... quadr. compl. ... <u>tetrahedral complex</u> ... complex ... Binet (Théorie des axes conjugués et des moments d'inertie des corps, 1811–13) ... H. Müller (1869) ...

<u>Theor. 1.</u> ... space Σ, Σ' ... collineation ... corresp. pt. ... li. complex ... tetrah. complex ... (Chasles, 1811)

Proof. ... coll. eq.

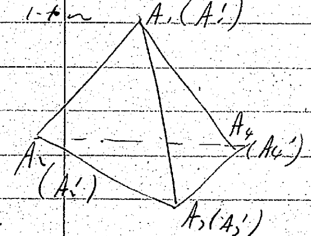
$x_1' = a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4$

$x_2' = a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4$

$x_3' = a_3x_1 + \ldots$

$x_4' = a_4x_1 + \ldots$

fundamental invariant tetrah. ; invariant tetr. ...



A_1 : $x_2 = x_3 = x_4 = 0$

A_1' : $x_2' = x_3' = x_4' = 0$

$a_2 = a_3 = a_4 = 0$

... $A_2 A_2'$, ...

$$\begin{cases} x_1' = a_1x_1 \\ x_2' = b_2x_2 \\ x_3' = c_3x_3 \\ x_4' = d_4x_4 \end{cases}$$

... coord. syst. ... x, x' ... li. ...

$$p_{12} = \begin{vmatrix} x_1 & x_1' \\ x_2 & x_2' \end{vmatrix} = \begin{vmatrix} x_1 & a_1x_1 \\ x_2 & b_2x_2 \end{vmatrix} = x_1x_2(b_2 - a_1)$$

$p_{34} = x_3x_4(d_4 - c_3)$, $\quad p_{14} = x_1x_4(d_4 - a_1)$

$p_{23} = x_2x_3(c_3 - b_2)$,

$$\therefore \frac{p_{12}p_{34}}{p_{14}p_{23}} = \frac{(b_2 - a_1)(d_4 - c_3)}{(d_4 - a_1)(c_3 - b_2)} = \text{const.}$$

... tetrah. comp. ...

Remark. Reye (1868) [illegible] synthetic [illegible] tetrah. cmpl. [illegible] Reye's Cmplx [illegible] part. case [illegible] Reye's Axen-kmplx [illegible] part case [illegible]

Thm. [illegible] rigid body [illegible] Σ, Σ' [illegible] com. pt. [illegible] line [illegible] tetr. cmplx [illegible] (Chasles)

[illegible] rigid body [illegible] any instant [illegible] direct. [illegible] tetr. cmp. [illegible] Bahn-kmplx [illegible] normal [illegible] linear cmplx [illegible] Bianchi [illegible] Jung, Geom. d. Massen (Enzykl. IV 1, p. 319) [illegible]

Line geom. [illegible]

Synth. method [illegible] Sturm, Liniengeometrie (Linear, quadr. cmplx)

Reye, II, III (linear cmp. tetrah. cmp.)

analyt. m.

Plücker, Neuen Geom.

Jessop, Line complex.

Koenigs, Geom. reglée

Zindler, Liniengeom. I, II, III [illegible]

Hudson, Kummer's quartic surface.

Lie, Berührungstransf.

Clebsch-Lindemann, II 1.

Diff. g. [illegible]

Bianchi, Dif. G. (Kummer, 1860, Sannia 1908) [illegible]

Timerding, Geom. d. Kräfte

Ball, Theory of Screws. Null-syst. [illegible] 20th. original [illegible]

Study, Geom. d. Dynamen

Schoenflies, Geom. d. Bewegung (synth.)

Koenigs, Leçons de cinematique

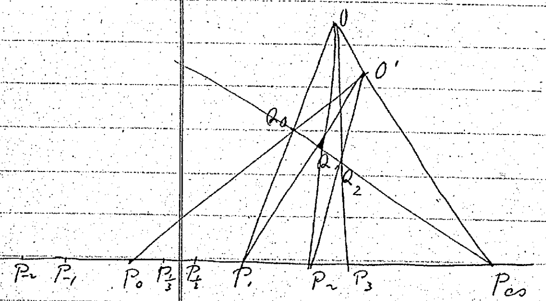
Cremona, Graphical statics. Null syst. [illegible] graph. statics [illegible] Maxwell (1864) [illegible] Cremona [illegible] (187-)

Chapter XVII. On the Foundation of Geometry

81. On the fundamental theorem of proj. geom.

[illegible] §13. [illegible] fund. theorem [illegible] §56 [illegible] metrically [illegible] metrical [illegible] (Klein, 1871—3) Klein, Nicht-Eukl. Geom. I, 338 ff.

[illegible] line [illegible] P_0 [illegible] P_∞ [illegible] P [illegible] O [illegible] OP_∞ [illegible] O' [illegible]



$\left.\begin{matrix} P_1 - O \\ P_0 - O' \end{matrix}\right\} Q_0 \quad \left.\begin{matrix} P_\infty - Q_0 \\ P_1 - O' \end{matrix}\right\} Q_1$

$\left.\begin{matrix} O - Q_1 \\ P_0 - P_\infty \end{matrix}\right\} P_2$

P_2 [illegible] $P_1 P_\infty$ [illegible] P_0 [illegible] harmonic conjugate [illegible]

$\left.\begin{matrix} P_2 - O \\ P_1 - O' \end{matrix}\right\} Q_1 \quad \left.\begin{matrix} P_\infty - Q_1 \\ P_2 - O' \end{matrix}\right\} Q_2 \quad \left.\begin{matrix} O - Q_2 \\ P_1 - P_\infty \end{matrix}\right\} P_3$

P_3 [illegible] $P_2 P_\infty$ [illegible] P_1 [illegible] h. conj. [illegible]

[illegible] $P_0, P_1, P_2, \dots, P_m, \dots P_\infty$

index [illegible] positive integer [illegible] pt. [illegible]

[illegible] $P_0 P_\infty$ [illegible] P_m $(m = 1, 2, \dots)$ [illegible] h. c. [illegible] P_{-m} [illegible]

[illegible] $P_{-1}, P_{-2}, \dots$ [illegible]

[illegible] $P_{\frac{1}{m}}$ [illegible] $P_0 P_1$ [illegible] P_∞ [illegible] h. c. [illegible] $P_{\frac{1}{2}}$ [illegible]

$P_0 P_{\frac{1}{2}}$ [illegible] P_1 [illegible] h. c. [illegible] $P_{\frac{1}{3}}$

$P_0 P_{\frac{1}{m}}$ [illegible] $P_{\frac{1}{m-1}}$ [illegible] h. c. $P_{\frac{1}{m+1}}$ [illegible]

[illegible] $P_{\pm \frac{p}{q}}$ [illegible] $P_0 \ P_1 \ P_\infty$ [illegible] $P_2, P_3, P_4 \dots$ [illegible]

$P_0 \ P_{\frac{1}{q}} \ P_\infty$ [illegible] $P_{1+\frac{1}{q}}, P_{2+\frac{1}{q}}, \dots$ [illegible]

$P_0 \ P_m \ P_\infty$ [illegible] P_{-m} [illegible]

$P_0 \ P_{\frac{p}{q}} \ P_\infty$ [illegible] $P_{-\frac{p}{q}}$ [illegible]

index [illegible] rational [illegible]

[illegible] pt range [illegible] continuity axiom (Dedekind's) [illegible] index number [illegible] continuity [illegible] P_α (α irrat.) [illegible]

[illegible] segment [illegible] length [illegible] parallel axiom [illegible]

[illegible] linear continuum [illegible] number continuum [illegible]

1 to 1 corresp. [illegible]

Def. [illegible] 4 pts $P_a \ P_b \ P_c \ P_d$ [illegible]

$\frac{c-a}{c-b} : \frac{d-a}{d-b}$ [illegible] double ratio [illegible]

$R(P_a P_b P_c P_d)$ [illegible]

$R_x(P_{a_1} P_{b_1} P_{c_1} P_{x_1}) = R_x(P_a P_b P_c P_x)$ [illegible]

$$\frac{c_1 - a_1}{c_1 - b_1} : \frac{x_1 - a_1}{x_1 - b_1} = \frac{c - a}{c - b} : \frac{x - a}{x - b}$$

$$\therefore x_1 = \frac{\alpha x + \beta}{\gamma x + \delta}$$

[illegible] proj. g [illegible] fund. theorem [illegible]

length [illegible] measurement [illegible] idea [illegible] number [illegible]

[illegible]

Theorem. Proj. g. [illegible] parallel axiom [illegible] independent [illegible] (Klein)

Remark [illegible] P_∞ [illegible] pt at ∞ [illegible]

[illegible] P

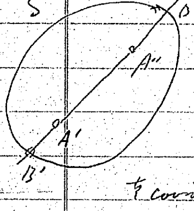
$P_0 \ P_2$ [illegible] Clebsch Lindemann II, p. 433.

[illegible] Killing, Grundlagen G. I p. 97 [illegible]
[illegible] Lüroth [illegible] (§ 13)

8. Projective measurements

plane [illegible] plane [illegible] fixed conic [illegible]

[illegible] eq. $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ (homog.)



[illegible] $A'(x'\,y'\,z')$

$A''(x''\,y''\,z'')$

$A'A''$ [illegible] conic [illegible] B', B''

[illegible] coord. $(lx'+mx'',\ ly'+my'',\ lz'+mz'')$ [illegible]

$l^2 S' + 2lmP + m^2 S'' = 0$ [illegible]

$S' \equiv ax'^2 + by'^2 + \cdots$

$S'' \equiv ax''^2 + by''^2 + \cdots$

$P \equiv ax'x'' + by'y'' + cz'z'' + f(y'z'' + y''z') + g(z'x'' + z''x') + h(x'y'' + x''y')$

$$R_x(A'A''B'B'') = \frac{P + \sqrt{P^2 - S'S''}}{P - \sqrt{P^2 - S'S''}}$$

[illegible] k [illegible] fixed constant [illegible]

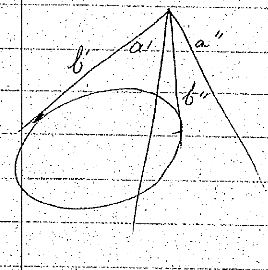
$$k \log \frac{P + \sqrt{P^2 - S'S''}}{P - \sqrt{P^2 - S'S''}}$$ [illegible] quality [illegible]

pts $A'A''$ [illegible] projective measurements [illegible] A', A'' [illegible] Distance [illegible] fixed conic [illegible] absolute conic [illegible] (Cayley, 1859)

[illegible] distance ambiguity ($B'B''$ [illegible]) [illegible] proj. [illegible] dist. [illegible] adequate [illegible] (Russell, p. 35) [illegible] irredu[illegible] [illegible] dist. [illegible] the metrical properties of space [illegible]

in duality fixed pt [illegible] line coord. [illegible]

(u, v, w), $\Sigma \equiv Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv$ [illegible]



$a'(u'\,v'\,w')$ $a''(u''\,v''\,w'')$

$a'a''$ [illegible] tangent [illegible]

b', b'' [illegible]

$(lu'+mu'',\ lv'+mv'',\ lw'+mw'')$

$l^2\Sigma' + 2lm\Pi + m^2\Sigma'' = 0$

$\Sigma' \equiv A'u'^2 + \cdots$ $\Sigma'' \equiv Au''^2 + \cdots$

$\Pi \equiv Au'u'' + Bv'v'' + Cw'w'' + F(v'w'' + v''w') + G(\ ) + H(\ )$

$$R_x(a'a''b'b'') = \frac{\Pi + \sqrt{\Pi^2 - \Sigma'\Sigma''}}{\Pi - \sqrt{\Pi^2 - \Sigma'\Sigma''}}$$

Σ [illegible] absolute conic $k \log \dfrac{\Pi + \sqrt{\Pi^2 - \Sigma'\Sigma''}}{\Pi - \sqrt{\Pi^2 - \Sigma'\Sigma''}}$ [illegible]

Def. [illegible] a', a'' [illegible] proj. [illegible] angle [illegible] absolute conic [illegible]

I. real conic [illegible] proj. meas. [illegible] hyperbolic [illegible]

II. [illegible] pts 2 (dist. [illegible]) [illegible] line (angl. [illegible]) degenerate [illegible] parabolic [illegible]

III. imaginary conic [illegible] elliptic [illegible]

[illegible] x, y [illegible] rectangular coord. [illegible] $z = 1$ [illegible]

$z = 0$ ... line at co ...

$S \equiv \varepsilon(x^2+y^2) - z^2 = 0$ ...

$\varepsilon > 0$ hyperbolic m.

$\varepsilon = 0$ $z^2 = 0$, ... parabolic m.

$\varepsilon < 0$ elliptic m.

... line coord ... S ...

$\Sigma \equiv u^2 + v^2 - \varepsilon w^2 = 0$

$\varepsilon > 0$ hyp.

$\varepsilon = 0$ par. circular pts at co ($u^2+v^2=0$) ...

$\varepsilon < 0$ ellip.

Klein 1871–72 ... measurements ... Euclidean g, hyperbolic g, elliptic g ...

<u>83. Euclidean geometry</u>

parabolic ... $k = -\frac{i}{2}$ ($\sqrt{-1} = i$)

$\Sigma' \equiv u'^2 + v'^2$

$\Sigma'' = u''^2 + v''^2$

$\Pi = u'u'' + v'v''$

$$\alpha = -\frac{i}{2}\log\frac{(u'u''+v'v'')+i(u'v''-u''v')}{(u'u''+v'v'')-i(u'v''-u''v')}$$

$$(u'u''+v'v'')+i(u'v''-u''v') = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

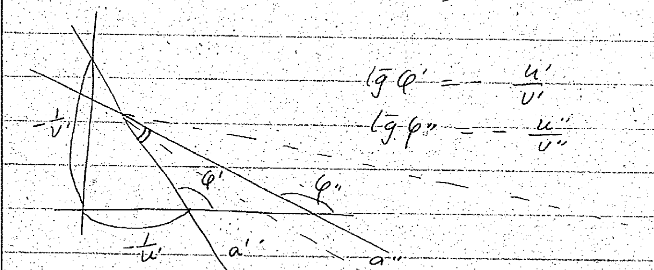
$$(\qquad) - i(\qquad) = r(\cos\theta - i\sin\theta)$$

$$\alpha = -\frac{i}{2}\log e^{2i\theta} = -\frac{i}{2}(2i\theta) = \theta$$

... $r\cos\theta = u'u'' + v'v''$

$r\sin\theta = u'v'' - u''v'$

$$\tan\theta = \frac{\left(-\frac{u''}{v''}\right) - \left(-\frac{u'}{v'}\right)}{1 + \left(-\frac{u'}{v'}\right)\left(-\frac{u''}{v''}\right)}$$



$\tan\varphi' = -\frac{u'}{v'}$

$\tan\varphi'' = -\frac{u''}{v''}$

circular pts at co

$\tan\theta = \tan(\varphi'' - \varphi')$

α ... a', a'' ... Euclidean angle ...

Laguerre's theorem (1853) ... circular pts at co, absolute ... proj measurements in angle ... angle ...

...

$K = -\frac{1}{2\sqrt{\varepsilon}}$ ... $\varepsilon = 0$ ... $\varepsilon = 0$ ...

$$D = \lim_{\varepsilon=0}\left\{-\frac{1}{2\sqrt{\varepsilon}}\log\frac{P+\sqrt{P-S'S''}}{P-\sqrt{P-S'S''}}\right\}$$

$S' \equiv \varepsilon(x'^2+y'^2) - z'^2$

$S'' = \varepsilon(x''^2+y''^2) - z''^2$

$P \equiv \varepsilon(x'x''+y'y'') - z'z''$

$$= \lim_{\varepsilon \to 0}\left\{-\frac{1}{2\sqrt{\varepsilon}}\log\left\{\frac{\varepsilon(x'x''+y'y'')-z'z''+\sqrt{\varepsilon}\sqrt{(x'z''-x''z')^2+(y'z''-z'y'')^2}}{\varepsilon(\quad)-z'z''-\sqrt{\varepsilon}\sqrt{\quad}}\right\}\right\}$$

$\sqrt{\varepsilon}$ [illegible] higher infinitesimal [illegible]

[illegible] binomial [illegible] expand [illegible]

$$= \lim\left[-\frac{1}{2\sqrt{\varepsilon}}\log\frac{-z'z''\left(1-\frac{\sqrt{\varepsilon}}{z'z''}\sqrt{\ }\right)}{-z'z''}\left\{1-\frac{\sqrt{\varepsilon}}{z'z''}\sqrt{\ }+\cdots\right\}\right]$$

$$= \lim\left[-\frac{1}{\sqrt{\varepsilon}}\log\left(1-\frac{\sqrt{\varepsilon}}{z'z''}\sqrt{\ }\right)\right]$$

$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \cdots$ [illegible] expand [illegible]

$$= \lim\left[-\frac{1}{\sqrt{\varepsilon}}\left(-\frac{\sqrt{\varepsilon}}{z'z''}\sqrt{\ }\right)\right]$$

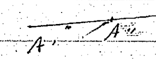
$$= \frac{\sqrt{\ }}{z'z''}$$

$$= \sqrt{\left(\frac{x''}{z''}-\frac{x'}{z'}\right)^2+\left(\frac{y''}{z''}-\frac{y'}{z'}\right)^2}$$

$z=1$, x, y rectangular coord [illegible]

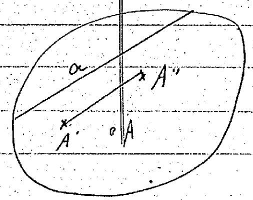
$$= \sqrt{(x''-x')^2+(y''-y')^2}$$

[illegible] length [illegible]



84 Hyperbolic geometry

absolute circle [illegible]



A', A'' [illegible] circle [illegible] proj. [illegible] distance

finite [illegible] A'' [illegible] circle

boundary [illegible]

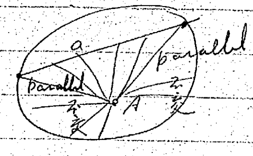
$S''=0$ [illegible] A'' [illegible] satisfy

$D = $ const [illegible]

[illegible] A' [illegible] boundary [illegible]

$S'=0 \quad D=\infty$

[illegible] absolute circle [illegible] proj. [illegible] pt at ∞ locus [illegible] lim a [illegible] A [illegible] a [illegible]



[illegible] Lobatschefskij-Bolyai [illegible] geom [illegible]

hyperbolic geom [illegible] plane [illegible] pt at ∞ [illegible] fixed circle [illegible] dist [illegible] absolute circle [illegible] proj. [illegible] dist [illegible] hyperbolic measurement [illegible] absolute circle [illegible] absolute [illegible] pt at ∞ [illegible] locus [illegible]

$$D = k\log\frac{P+\sqrt{\ }}{P-\sqrt{\ }}$$

$$e^{\frac{D}{2k}} = \sqrt{\frac{P+\sqrt{P^2-S'S''}}{P-\sqrt{\ }}}$$

$$e^{-\frac{D}{2k}} = \sqrt{\frac{P-\sqrt{\ }}{P+\sqrt{\ }}}$$

$$\therefore \frac{e^{\frac{D}{2k}} - e^{-\frac{D}{2k}}}{2} = \sqrt{\frac{P^2 - s's''}{s's''}}$$

[illegible] $\frac{D}{2ki} = i\sqrt{\frac{P^2 - s's''}{s's''}}$

[illegible] hyperbolic plane [illegible]

(x, y, z) $x+dx,\ y+dy,\ z+dz$ $z=1,\ dz=0$

[illegible] distance ds [illegible]

expand [illegible]

$$\frac{ds}{2ki} - \frac{1}{3!}\left(\frac{ds}{2ki}\right)^3 + \cdots = i\sqrt{\frac{[\varepsilon\{x(x+dx)+y(y+dy)\} - z(z+dz)]^2 - SS'}{[\varepsilon(x^2+y^2) - z^2][\varepsilon\{(x+dx)^2+(y+dy)^2\} \ldots]}}$$

$$S = \varepsilon(x^2+y^2) - z^2$$

$$S' = \varepsilon[(x+dx)^2 + (y+dy)^2] - (z+dz)^2$$

$$P = \varepsilon[x(x+dx) + y(y+dy)] - z(z+dz)$$

higher order infinitesimal [illegible]

$$\frac{ds^2}{4k^2} = \frac{[\varepsilon\{\quad\quad\} - z(z+dz)]^2 - SS'}{[\quad\quad][\quad\quad]}$$

$$= \frac{\varepsilon[(xdz - zdx)^2 + (ydz - zdy)^2]}{\{\varepsilon(x^2+y^2) - z^2\}^2}\left[1 + \frac{\varepsilon(xdx+ydy) - zdz}{\varepsilon(x^2+y^2) - z^2}\right]^{-1}$$

[illegible] binomial [illegible] expand [illegible]

$$\frac{ds^2}{4k^2} = \frac{\varepsilon[(xdz - zdx)^2 + (ydz - zdy)^2]}{\{\varepsilon(x^2+y^2) - z^2\}^2}$$

$\therefore z = 1,\ dz = 0$ [illegible] Cartesian coord [illegible]

$$ds^2 = 4k^2\varepsilon\,\frac{dx^2+dy^2}{[\varepsilon(x^2+y^2) - 1]^2}$$ [illegible] Euclid [illegible] $ds^2 = dx^2 + dy^2$

[illegible] differential quadratic form [illegible] Krümmungsmass [illegible] $-\frac{1}{k^2}$ [illegible]

[illegible] curvature [illegible] negative constant [illegible]

Dif. geom. [illegible] line element [illegible] surface

curvature negative constant [illegible] surface [illegible]

hyp. plane [illegible] negative const. curvature [illegible]

surface [illegible] deformable [illegible] deformation

[illegible] plane [illegible] surface [illegible] geodesic

line [illegible] hyp. plane [illegible] negat.

const. surface [illegible] (Beltrami, 1866)

negat. const. curv. surface [illegible] pseudo-spherical surface

[illegible] (ds [illegible])

[illegible] hyp. plane [illegible] pseudo-sph. surface

[illegible] deformable [illegible] pseudo-sph. surface

[illegible] singular pt, singular line

[illegible] (tangent [illegible] antipross) [illegible]

[illegible] pt) [illegible] singular pt [illegible]

[illegible] hyp. plane [illegible]

[illegible] Hilbert (1901) Holmgren (1902)

85. Elliptic geometry

absolute curve [illegible] imaginary [illegible] $\varepsilon < 0$

[illegible] real pt [illegible] $S = 0$ [illegible]

Thus it happens that geodesics, though in the plane they have only one pt in common, may on the cylinder have an infinite no. of intersections. Somewhat similar to this is the rel. betw. the spherical + ellipt. Geometries. To any one pt in ellipt. space, two pts correspond in spherical space. Thus geodesics, which in spherical space may have two pts in common, can never, in ellipt. space, have more than one intersection (Russell p. 39 vgl. Kl. 2, 97 ff. Nicht-Eukl. Bd. 1, 292 ff)

$S' \neq 0$, $S'' \neq 0$ [illegible]

[illegible] $\log \frac{P+\sqrt{\ }}{P-\sqrt{\ }}$ [illegible] finite [illegible]

[illegible] finite [illegible] pt at [illegible] exist [illegible] Helmholtz [illegible] Russell 38

[illegible] finite [illegible] parallel [illegible]

Elliptic gen. Riemann space [illegible]

pt. order [illegible] space proj. g. [illegible]

[illegible] Axiom der Verknüpfung [illegible]

[illegible] A', A'' dist. [illegible]

imag. [illegible] B', B'' [illegible] imaginary conjugate imag. [illegible]

[illegible] pt [illegible] a harmonic ratio [illegible] pure imag. [illegible]

$$D = iK' \log \frac{P+\sqrt{P^2-S'S''}}{P-\sqrt{P^2-S'S''}}$$ [illegible] dist

real [illegible]

$$ds^2 = -4K'^2 \epsilon \frac{dx^2+dy^2}{[\epsilon(x^2+y^2)-1]^2}$$ [illegible]

$\epsilon < 0$ [illegible] $ds^2 > 0$

[illegible] dif. quadratic [illegible] + $\frac{1}{K^2}$ [illegible]

posit. constant [illegible] surface

[illegible] const. curv. + $\frac{1}{K^2}$ [illegible] surface [illegible] ellipt.

plane spherical surf. deformable [illegible]

[illegible] surface [illegible] geodesic [illegible]

~~spheres~~ [illegible] ellipt. pl. [illegible]

deformable [illegible] surface sphere

[illegible] (Liebmann, 1899)

[illegible] space [illegible] 2 R [illegible]

[illegible]

absolute [illegible] invariant [illegible] proj.

coll. [illegible] proj. dist. + angle invariant [illegible]

[illegible] absol. [illegible] coll. [illegible] non-Euclid.

plane [illegible] define its proj. g. [illegible]

[illegible] Non-Euclid. [illegible] Kl. [illegible]

Nicht-Euklid. G. Klein - Elliptische automorphe

Funkt. ([illegible]) Coolidge [illegible] Eucl.

g. [illegible] last [illegible] chapter [illegible]

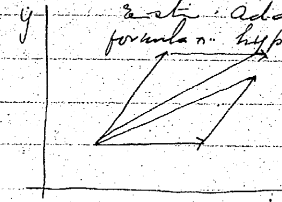
Non-Eucl. g. [illegible] math. [illegible]

[illegible] Kl. [illegible] Number theory [illegible]

automorph. f. [illegible] apply [illegible] B. [illegible]

Relativity [illegible] velocity component $\dot{x}\,\dot{y}$ [illegible] Cartesian

coord. [illegible] Borel [illegible] kinematical space [illegible]

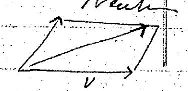


[illegible] addit. formula [illegible] hyp. triangle [illegible]

[illegible] space [illegible] hyp. space

[illegible] Varičak

(Servian) [illegible] (1910-1912)



physical space: [illegible] space [illegible] 3.9
[illegible] 2 [illegible] 0 [illegible] (aberration [illegible])

All the fruitful uses of imaginaries, in G., are those which begin & end with real quantities, & use imaginaries only for intermediate steps. ---- If the quantities with which we end are capable of spatial interpretation, then, & only then, our result may be regarded as geometrical. To use geom. language in any other case, is only a convenient help to the imagination. To speak, for example, of proj. properties which refer to the circular points, is a mere *memoria technica* for purely algebraical properties; the circular pts are not to be found in space, but only in the auxiliary quantities by which geom. equations are transformed. That no contradictions arise from the geom. interpretation of imaginaries, is not wonderful: for they are interpreted solely by the rules of Algebra, which we may admit as valid in their application to imaginaries. Russell, Found. of G. 45.

In any world in which perception presents us with various things, with discriminated & differentiated content, there must be, in perception, at least one "principle of differentiation" (Bradley's Logic p. 63), an element, that is, by which the presented things are distinguished as various. This element, taken in isolation, & abstracted from the content which it differentiates, we may call a form of externality. That it must, taken in isolation, appear as a form, & not as a mere diversity of material content, is, I think, fairly obvious. For a diversity of mat. content cannot be studied apart from that mat. content; what we wish to study here, on the contrary, is the bare possibility of such diversity, which forms the residuum, as I shall try to prove hereafter (p. 182 ff), when we abstract from any sense-perception all that is distinctive of its particular matter. This possibility, then, this principle of bare diversity, is our form of externality.
136

A straight line, then, is not the *shortest* distance, but is simply *the* distance between two pts — so far, this conclusion has stood firm (two pts = unique … straight line … ex. 152). But suppose we had two or more curves through two pts, & that all these curves were congruent *inter se*. We should then say, in accordance with the def. of spatial equality, that the lengths of all these curves were equal. Now it might happen that, although no one of the curves was uniquely determined by the two end-points, yet the common length of all the curves was so determined. In this case, what would hinder us from calling this common length the distance apart, although no unique figure in space corresponded to it? This is the case contemplated by spherical Geom., where, as on a sphere, antipodes can be joined by an infinite no. of geodesics, all of which are of equal length. The difficulty is, therefore, not a purely imaginary one, but one which modern Geom. forces us to face. 168

It will be remembered that, in our a priori proof that two pts must have one definite rel., we held it impossible for those two pts to have, to the rest of space, any rel. which would be unaltered by motion. Now in spherical space, in the particular case where the two pts are antipodes, they *have* a rel. unaltered by mot., to the rest of space — the rel., namely, that their distance is half the circumference of the universe. In our former discussion, we assumed that any rel. to outside space must be rel. of position — & a rel. of posit. must be altered by mot. But with a finite space, in which we have absolute magnitude, another rel. becomes possible, namely, a rel. of magnitude. Antipodal pts, accordingly, like coincident pts, no longer determine a unique straight line. And it is instructive to observe that there is, in consequence, an ambiguity in the expression for distance, like the ordinary ambiguity in angular measurement. If $1/k^2$ be the space constant, & d be one value for the dist. between two pts, $2\pi kn \pm d$, where n is any integer, is an equally good value 169 / Dist. is, in short, a periodic funct. like angle. Thus such a state of things rather confirms than destroys my contention, that dist. depends on a curve uniquely determined by two points. For as soon as we drop this unique determination, we see ambiguities creeping into our expression for dist. Dist. still has a set of *discrete* values, corresponding to the fact, that, given one pt., the straight line is uniquely determined for all other pts but one, the antipodal pt. It is tempting to go on, & say: If through every pair of points there were an infinite no. of the curves used in measuring dist., dist. would be able, for the same pair of pts, to take not only a discrete series, but an infinite continuous series of values.

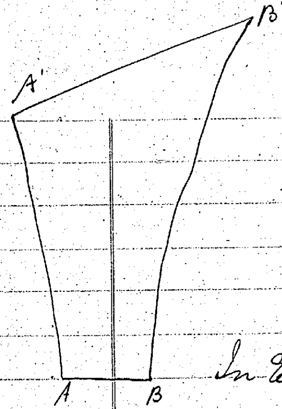
This, however, is mere speculation. I come now to the *pièce de résistance* of my argument. The ambiguity in spherical space arose, as we saw, from a rel. of *magnitude* to the rest of space — such a rel. being unaltered by a motion of the two pts, & therefore falling outside our introductory reasoning. But what is this rel. of magnitude? Simply a rel. of the *dist.* between the two pts to a *dist.* given in the nature of the space in question. It follows that such a rel. *presupposes* a measure of dist., & need not, therefore, be contemplated in any argument which deals with the *a priori* requisites for the possibility of definite distances.

I have now shown, I hope conclusively, that spherical space affords no object to the apriority of my axiom. Any two points have one rel., their dist., which is independent of the rest of space, & this rel. requires, as its measure, a curve uniquely determined by these two pts. I might have taken the bull by the horns, & said: Two pts *can* have no rel. but what is given by lines which join them, & ∴, if they have a rel. independent of the rest of space there must be one line joining them which they completely determine. Thus James says (Psych. II. 149–150): 170/

"Just as, in the field of quantity, the rel. between two nos is another no., so in the field of space the rel.s are facts of the same order with the facts they relate. . . . When we speak of the rel. of direction of two pts towards each other, we mean simply the sensation of the line that joins the two pts together. The line is the rel. . . . The rel. of ~~direct~~ position between the top & bottom pts of a vertical line is that line, & nothing else." 171

Measurement, we may say, is the application of no. to continua, or, if we prefer it, the transformation of mere quantity into no. of units. Using *quantity* to denote the vague more or less, & *magnitude* to denote the precise no. of units, the problem of measurement may be defined as the transformation of quantity into magnitude.

Now a no., to begin with, is a whole consisting of smaller units, all of these units being qualitatively alike. In order, therefore, that a continuous quantity may be expressible as a number, it must, on the one hand, be itself a whole, & must, on the other hand, be itself divisible into qualitatively similar parts. In the aspect of a whole, the quantity is _intensive_; in the aspect of an aggregate of parts, it is _extensive_. A purely intensive quantity, ∴, is not numerable — a purely extensive quant. if any such could be imagined, would not be a single quant. at all, since it would have to consist of wholly unsynthesized particulars. A measurable quant., ∴, is a whole divisible into similar parts. But a continuous q., if divisible at all, must be _infinitely_ divisible. For otherwise the pts at which it could ~~be~~ be divided would form natural barriers, & so destroy its continuity. But further, it is not sufficient that there should be a possibility of division ~~of~~ into mutually external parts; while the parts, to be perceptible as parts, must be mutually external parts; while the parts, to be perceptible as parts, must be mutually external, they must also, to be knowable as _equal_ parts, be capable of overcoming their mutual externality. For this, as we have seen, we require superposition, which involves free Mobility & homogeneity — the absence of free M. in time, where all other requisites of measurement are fulfilled, renders direct measurement of time impossible. Hence infinite divisibility, free Mob., & homogeneity are necessary for[176] the possibility of measurement in _any_ continuous manifold, & these, as we have seen, are equivalent to our three axioms. These axioms are necessary, ∴, not only for spatial measurement, but for all measurement. The only manifold given in experience, in which these conditions are satisfied, is space. 177



Coolidge 48–53

$AA' = x$

$$\lim_{AB=\infty} \frac{m\ A'B'}{m\ AB} = \phi(x)$$

(measure)

$$\cos\frac{x}{k} = \phi(x) = 1 - \frac{x^2}{k^2 2!} + \frac{x^4}{k^4 4!} - \cdots$$

$\frac{1}{k^2}$ measure of curvature of space

In Euclidean space (parabolic)

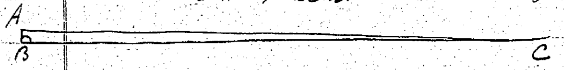
$\frac{1}{k^2} = 0 \quad k = \infty$

$\cos\frac{x}{k} = \cos 0 = 1 = \phi(x)$ constant

Hyperbol. sp. $\frac{1}{k^2} < 0$ Ellipt. sp. $\frac{1}{k^2} > 0$

$k' = ik$, $\phi(x) = \cosh\left(\frac{x}{k'}\right)$

BAC, const. $\theta = \angle BAC$



$$\lim \frac{AB}{AC} = f(\theta) = \cos\theta$$

a, b, c measures of opposite sides of ABC

$\cos\frac{b}{k} = \cos\frac{a}{k}\cos\frac{c}{k}$ expand in ∴

Euclid case $b^2 = a^2 + c^2$, Pythagorean th.

Euclid — Axiom [illegible] postulate [illegible] common not. [illegible]

5. post. parallel-postulate (Axiom 11 or 12 [illegible])

The measurement of angles is independent of the theory of parallels. Vertically opposite angles are equal; the sum of the four angles made by two intersecting ~~angles~~ lines is an absolute constant, & one quarter of this is a right angle. An absolute unit of angle, therefore, exists. A "flat-angle", which is equal to two right angles, is generally denoted by the symbol π. Through a given pt. only one perpendicular can be drawn to a given straight line, the usual construction for this being always possible. But it is necessary to warn the reader that π does not stand for the ratio of the circumference of a circle to its diameter, for in non-Eucl. geom. this ratio is not constant; & the radian, unit angle, in terms of which a flat angle is represented by the no. π, does not admit of the familiar construction by means of a circle.

Sommerville, 28.

ref. below.

3 cases in hyp. G. (two straight lines)

(1) Intersecting, & have a real angle of intersect. but no common perpendicular.

(2) Non-intersecting, & have a real real shortest dist or common perpendicular, but no real angle.

(3) parallel, with a zero angle & zero shortest dist or common perp. at ∞.
(equidistant lines. &c.)

In hyp. G. on every line there are two pts at ∞, & the assemblage of pts at ∞ in a plane is a curve of the 2nd degree or conic, since it is met by any line in two pts. In three dim. the assemblage is a surface of the 2nd degree or quadric. This figure, the assemblage of all the pts at ∞, is called the Absolute.

When two pts at ∞ approach coincidence, the line determined by them becomes a tangent to the absolute. As such a line is called a line at ∞. Similarly we obtain planes at ∞, which are tangent planes to the Absolute.

In Euclid. G. there is just one parallel through a given pt to a given line in a plane, & the two pts at ∞ upon a line coincide. The assemblage of pts at ∞ in a plane thus reduces to a double line, the line at ∞, which is a degenerate case of a conic. There is in this case only one real line at ∞, but any line whose equat. in rectangular coord. is of the form $x \pm iy + c = 0$ is at an infin. dist. from the origin, since $1 + i^2 = 0$, & the assemblage of these lines consists of two imag. pencils. The equat. of the line at ∞ is $\alpha \equiv 0x + 0y + 1 = 0$ & the equat. of the two pencils are $\omega + \lambda\alpha = 0$, $\omega' + \lambda\alpha = 0$, where $\omega, \omega' = x \pm iy$.

The absolute in Eucl. G. thus consists, as a locus of pts, of the line at ∞ $\alpha = 0$ taken twice, &, as an envelope of lines, of two imag. pencils $\omega + \lambda\alpha = 0$ $\omega' + \lambda\alpha = 0$, with their vertices on the line at ∞. These two imag. vertices are the pts of intersect. of the pt-circle $\omega\omega' \equiv x^2 + y^2 = 0$ with the line at ∞. Since the equat. of any circle can be written $\omega\omega' + u\alpha = 0$, where $u = 0$ represents a straight line, we see that every circle passes through the two pts ($\omega\omega' = 0$, $\alpha = 0$), & for this reason these two imag. pts are called the circular pts.

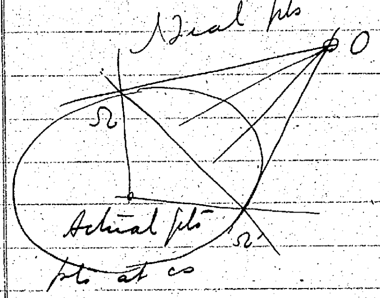
In Eucl. G. of 3 dim. the absolute consists:

general equat. of circle. $x^2 + y^2 + z(ax + by + cz) = 0$
$z = 0$ (line at ∞) & $x + iy = 0$, $x - iy = 0$ ∴ pair of imag. lines
3 ... circle ...

as a locus of pts, of the plane at ∞ taken twice, & as an envelope of planes, of all the planes through tangents to an imag. circle, the intersection of the pt-sphere $x^2+y^2+z^2=0$ with the plane at ∞.
46 — 47

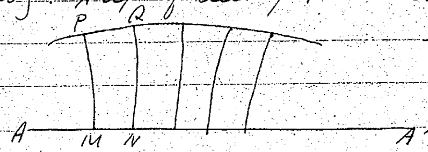
If a system of lines is such that any two are coplanar, while they do not all lie in the same plane & are neither parallel nor intersect, then they are all perpendicular to a fixed plane.

We shall call this system, which is completely determined by two of the lines, or by a certain plane π, a bundle of lines with an *ideal* vertex O.
π axis.



all the lines of the pencil with vertex O ⊥ ΩΩ'

circle = locus of pts at a constant dist from a fixed pt.
A circle cuts all its radii at ⊥, orthogonal trajectory of a pencil of lines with a real vertex.
Let the vertex go to ∞; then the lines of the pencil become ∥, & the circle takes a limiting form, which is not, as in ordinary geom., a straight line but is a uniform curve. This curve, a circle with inf. radius, is called a *horocycle*; it is the orthogonal trajectory of a pencil of parallel lines. All horocycles are superposable.
orthog. traj. of lines of pencil with ideal vertex — equidistant curve.



circle: ≡ 3 { circle — sphere; horocycle — horosphere; equi-dist. c — equi-dist. surf.

plane sect. of sphere — circle
of horosph. — circle, except when the sect. is normal to the surface, then horocycle.
plane ∥ axial pl. 3 ... circle
of equi-dist surf. — ... ... 14 equidist. curve.

Geom. of a bundle of lines + planes.

In plane g we have pts, lines, dist. + angles; in a bundle of lines + planes through a pt O we have lines, planes, plane agls + dihedral agls. Let us change the language to make it resemble the lang. of planes. In translating from one lang. to another we require a dictionary. The following will suffice.

"Point" - - - Line through O
"Line" Plane " O
"Dist" betw. two pts" Angle betw. two lines through O
"Angle" betw. two lines Dihedral angle betw. two planes thr. O
"Parallel lines" Parallel planes

The two "pts" determine a "line" + two "lines" determine a "pt". "Parallel lines" only exist when O is at ∞ or ideal.

When O is at ∞, through a given "pt" there passes just one "line" "parallel" to a given "line"; + when O is ideal, two "parallels" can be drawn through a given "pt" to a given "line".

There are three kinds of geom. of a bundle according as the vertex O is actual, at ∞ or ideal, + these are exactly of the same form as elliptic, parabolic (i.e. euclid.) + hyperbolic plane geom.

If a sphere be drawn with centre O cutting the lines + planes of the bundle, we can get a further correspondence. When O is an actual pt we have a proper sphere. We have the the follow. dictionary:

"Point" Pair of antipodal pts on sphere
"Line" Great circle
"Dist" Length of arc
"Angle" betw. "lines" Angle betw. great circles

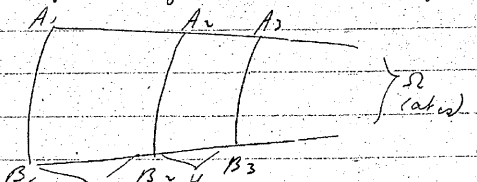
Hence the geom. on a proper sphere, where great circles represent lines, + pairs of antipodal pts represent pts, is the same as ellipt. g.

When O is ideal, the sphere becomes an equidist. surface, + its geom. is hyperbolic, when O is at ∞ it becomes a Horosphere, + its geom. is euclidean "pt" in each case being represented by a pt, + "lines" by normal sections, which are also shortest lines or geodesics on the surface.

We have here the important + remarkable theorem that <u>the geom. on the horosphere is euclid.</u>

57 — 56

Ratio of arcs of concentric horocycles



$$\frac{A_1B_1}{A_2B_2} = f(x) \qquad \frac{A_2B_2}{A_3B_3} = f(y)$$

$$\frac{A_1B_1}{A_3B_3} = f(x+y) = f(x)f(y)$$

$$\therefore f(x) = c^x \qquad c \text{ absolute constant} > 1$$

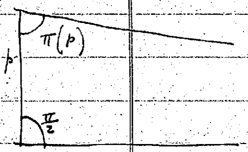
Helmholtz [illegible] Riemann, [illegible] ds^2 (line element ds) [illegible] dx^2, dy^2, dz^2 homog. eq. of 2nd deg. [illegible]
(Riemann: $ds = \sqrt{\Sigma dx^2} / (1 + \frac{1}{4}\alpha \Sigma x^2)$ $\alpha = \frac{1}{k^2}$ [illegible] + c)
(free mobil. 1. monodromony [illegible]) Lie. Helm. [illegible] (motion) [illegible] transform. group [illegible]

$f(x)$, pure ratio $\log f(x)$ or $x \log c$ pure ratio

$\log_e c = \frac{1}{k}$ $f(x) = e^{\frac{x}{k}}$

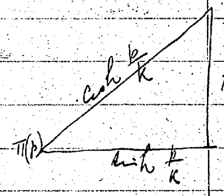
k, absolute linear const. space-constant

k is the length of the arc of a horocycle which is such that the tangent at one extremity is ∥ to the radius through the other extremity 57–58

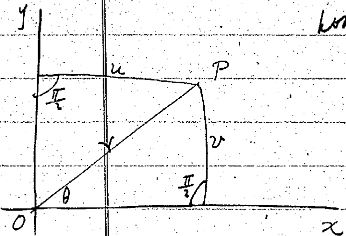


$\tan \frac{1}{2}\Pi(p) = e^{-\frac{p}{k}}$

$\cot \Pi(p) = \sinh \frac{p}{k}$



(60–61)



homogeneous coord.

$x = k \sinh \frac{u}{k} = k \sinh \frac{r}{k} \cos\theta$

$y = k \sinh \frac{v}{k} = k \sinh \frac{r}{k} \sin\theta$

$z = \cosh \frac{r}{k}$

Weierstrass' point-coord.

(hyp. [illegible] k [illegible] ik [illegible])
$x^2 + y^2 - k^2 z^2 = -k^2$ [illegible]

$x^2 + y^2 - k^2 z^2 = 0$ real absolute
$+ k^2 z^2 = 0$ imag. absol. ellip. g.

ideal pts (outside the absolute), ultrainfinite

4 Periods of Non-Eucl. G.

1. synth. elem. M. Gauss. Lobat. Bolyai
2. geodesic repr. Riem. Helm. Lie. Beltrami
3. proj. repr. Cayley. Klein
4. logical foundy. (Staudt) Pasch, Hilbert, Peano, Pieri, Veblen.

Starting with the group of proj. transfs., Lie determines the character of the transformation so that this axiom of free mobility in the infinitesimal may be verified & he proves that they form a group which leave unaltered either a non-ruled surface of the 2nd degree (real or imag. ellipsoid, hyperboloid of two sheets or ellip. paraboloid), or a plane & an imaginary conic lying on ~~this~~ plane. This invariant figure is just the Absolute. The motions of space, form a sub-group of the general proj. group of pt-transfs which leave the Absolute invariant. And so, without ~~[illegible]~~ Helmholtz's axiom of monodromony but using a definite assumption of free mobility, Lie establishes that the only possible types of metrical geom. are the 3 ~~types~~ in which the Absolute is a real non-ruled quadric (hyp. g.), an imaginary quadric (ellipt. g.), & a plane with an imag. conic (Eucl. g.) 198–199

Assumpt. of coord.

There are several pts on which the investigat.s of Riem. Helm. & Lie admit of criticism. The outstanding difficulty which strikes one at the once lies in the use of coord. How can we define the coord of a pt. before we have fixed the idea of congruence? This quest. has been settled by an appeal to the famous procedure of Von Staudt. He has shown (Geom. d. Lage, & Beiträge z. G. d. L.) how, by means of repeated applicat. of the quadrilateral construct. for a harmonic range, no.s ~~can~~ may be assigned to all the pts of a line. This, & other questions involved, have now been solved by the modern procedure of Pasch, Hilbert & the Italian school represented by Pieri. This procedure, which marks a return to the classical method of Euclid, consists in developing geom. as a purely logical system deduced from an appropriately chosen system of axioms or assumpt.s. 199

space. dim. – ... axiom ... = ...

no. of dim. & curvature of space is q[uan]titative orig.

While the concept. of non-Eucl. space of 3 dim. in no way implies necessarily space-curvature of or a fourth dim., it is still an interesting speculat. to suppose that we exist really in a space of 4 dim. but with our experience confined to a certain curved locus in this space, just as Helmholtz's "two-dimensional beings" were confined to the surface of a sphere in space of three dim.s, & acquired in this way the idea that their geom. is non-Eucl.

W. K. Clifford (The Common Sense of the Exact Science chap. iv. 8. (9)) has gone further than this & imagined that the phenomena of electricity &c. might be explained by periodic vibrat.s in the curvature of space. But we cannot now say that this three-dim. universe in which we have our experience is <u>space</u> in the old sense, for space, as distinct from matter, consists of a changeless set of terms in changeless rel.s. There are two alternatives. We must either conceive that space is really of four dim.s & our universe is an extended sheet of matter existing in this space, the aether if we like; & then, just as a plane surface is to our 3-dimensional intelligence a pure abstract., so our whole universe will become an ideal abstract. existing only in a mind that perceives space of four dim — an argument which has been brought to the support of Bishop Berkeley ((Hinton, Scientific Romances (series p. 31; — the 4th Dim.). Or, we must resist our innate tendencies to separate out space & bodies as distinct entities, & attempt to build up a monistic theory of the physical world in terms of a single set of entities, material pts, conceived as altering their rels with time (Whitehead, On math concepts of the mat. world). In either case it is not space that is altering its qualities, but matter which is changing its form & rels with time. 200–201

Lobat. geom. Eucl. representation ... Beltrami ... geom. consistency ... absolute test of consistency ... 202

Pure visual space, which is the limited field of our imaginary one-eyed sessile philosopher, is a crude elliptic two-dim space 208

These astronomical experiments are based upon the received laws of astronomy & optics, which are themselves based upon the euclidean assumption. It might well happen, then, that a discrepancy observed in the sum of the angles of a triangle could admit of an explanation by some modification of these laws, or that even the absence of any such discrepancy might still be compatible with the assumption of non-euclid. geom. (paradox) 209

"All measurement involves both physical & geom. assumptions, & the two things, space & matter, are not given separately, but analyzed out of a common exp. Subject to the general condition that space is to be changeless & matter to move about in space, we can explain the same observed results in many different ways by making compensatory changes in the qualities that we assign to space & the qualities we assign to matter. Hence it seems theoretically impossible to decide by any experiment what are the qualities of one of them in distinction from the other." (Broad, ...) 209–10

The conclusion thus arrived at by Poincaré is quite akin to the modern doctrine in physics expressed by the Principle of Relativity. Just as, according to this doctrine, it is impossible by any means to obtain a knowledge of absolute motion, so according to Poincaré, it is beyond our power to obtain a knowledge of absolute space.

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