NOTE BOOK

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Liebmann, Zur Analyse der WaK III Aufl.

Nach Alledem wird man nun zu der Frage gedrängt, ob es für die intuitive Notwendigkeit des Euklidischen Raums irgendeine Erklärung gibt. Dies aber lässt eine doppelte Deutung zu. Denn man versteht einmal unter "Erk." bloss die log. Angabe der wesentlichen Merkmale, welche schon dann erreicht ist, wenn man den Specialfall durch Determination des genus proximum mittelst der specifischen Differenz gegen andere, ihm coordinierte Specialfälle begrifflich ~~be~~abgrenzen kann. Ausserdem wird jedoch unter "Erk." die Deduction der Tatsache als notwendige Folge aus dem zureichenden Grunde verstanden, wie dies in allen Realwissenschaften geschieht. Die erste Art von "Erk." wird nun ~~in~~ allerdings in der Metageometrie geliefert, die zweite aber nicht. Metageometrie ist analytische Raumdefinition oder, wenn man will, Raumcharakteristik, aber nicht causale R-deduction. 80 R-char. P.

Hier kommt denn nun die theoret. Notwendigkeit der Newtonschen „absoluten Z." zu Tage. Sie ist ein theoret. Machtspruch, ein Postulat d. math. Vernunft. Will man nicht allen Boden unter den Füssen verlieren, will man nicht unsere phoronomischen Fundamentalbegriffe, Galileis lex inertiae, u. die bekannten apriorischen Relationen zw. R, Z, Geschwindigkeit, Beschleunigung, auf denen unsere gesamte math. Naturphil. beruht, über den Haufen werfen, so sieht man sich, unter Abstrakt. von jedem emp. Maassprincip, genötigt zur Idee einer von allem Wechsel, von der Unregelmässigkeit so gut des äusseren wie des inneren Geschehens völlig emancipirten absoluten Z., welche „schlechthin gleichmässig dahinfliesst (quod aequabiliter fluit)", wie Newton sagt. Diese absol. Z. bildet mit der absol. Beweg. u. dem absol. R (d.h. einem fundamentalen, unbeweglichen System von drei Weltachsen) eine Trias unentbehrlicher

Hypothesen, theoretischer Grundideen, auf welchen das ganze, feingegliederte Lehrgebäude der math. Naturwiss. beruht. Hierin, aber hierin allein, liegt die Berechtigung des Begriffs jener absol. Z.; in ihrer theoret. Unentbehrlichkeit. 104 sub., obj. u. absol. Z.

absol. Z. [illegible] a priori [illegible] in Axiomen der Chronometrie [illegible] Ubiquität [illegible] (Homogenität) [illegible] 104–5.

Wer das Trägheitsgesetz anerkennt, der gibt absolute Beweg. zu; u. wer diese leugnet, stösst jenes um. Was behauptet jenes Gesetz? „Ein durchaus sich selbst überlassener Körper bewegt sich mit der Geschwindigkeit, die er gerade besitzt, u. in der Richtung, die er augenblicklich verfolgt, sich selbst ins Unendliche weiter." Durchaus überlassen ist aber der Körper nur dann, wenn man von allen ~~übrigen~~ andern ~~mit jenem~~ Körpern gänzlich abstrahirt (u. ihn allein, isolirt im leeren R ~~im R coexistirenden Dinge völlig kraftlos u. einflusslos sein, sie werden~~

~~anziehend noch zurückstossend, weder accelerierend o. retardierend, weder durch actio in distans ablenkend noch contact vermöge~~ sein spiel treiben lässt 137 Rel. u. absol. Beweg.

„Ein sich selbst überlassener Körper läuft geradlinig in Beziehung auf den absoluten R." — Dies will das Trägheitsgesetz besagen. 138

Woher stammt ~~dieses~~ einerseits das subj. Notwendigkeitsleben, welches jenen apodiktischen Gesetzen anhaftet? Wie erklärt sich andrerseits der Umstand, dass eben die obj. Erf. niemals widersprechen kann? Wer dies Problem noch nicht erfasst hat, dem fehlt die letzte Weihe der Phil., der steht als Neophyt noch draussen vor der Tür; so z. B. Herder mit seiner Metakritik, diesem deplorablen Machwerk. — Und hier ist es denn mindestens eine statthafte Hypothese: „Unsere Intelligenz wird von höchsten Gesetzen beherrscht, denen sowohl unsere intellektuelle Imagination, als unsere

Wahrnehmungsintelligenz gehorcht, aus denen die mehrgenannten Grundwahrheiten hervorgehen, u. welchen die emp. Wirklichkeit aus dem Grunde unweigerlich entspricht, weil sie eben nur ein Phänomen innerhalb unserer wahrnehmenden Intelligenz ist, daher den Gesetzen derselben unterworfen ist, wie das Phänomen der wechselnden Figuren im Kaleidoscop den in der Construction dieses Instrumentes begründet liegenden Gesetzen." Ich kenne wenigstens keine andere Hypothese, die mit dieser rivalisiren kann 256 Die Metamorphose des apriori.

Tatsächlich bedient sich aber die moderne Physik auf Schritt u. Tritt, d. Hypothesen, auch dann, wenn sie es zu leugnen sucht (und zwar sind ihre Hypothesen vierfacher Art 1 solche der Abstraktion u. Begriffsbildung, 2 solche der Gesetzesinduktion, 3 solche der Ursacheninduktion u. 4 erkenntnistheoretische Hypothesen über die Existenz u. nähere Beschaffenheit einer realen Natur 208

Der methodol. u. erkenntnistheoret. Standpunkt der modernen Physik

Daß es eine wirkliche Natur gibt, und daß die von der Physik aufgestellte Gesetze in dieser wirklichen Natur Geltung haben, ist selbst nur eine Hypothese. Denn die wirkliche Natur ist ja nichts als die hinter der erschlossene hypothetische Ursache unserer subjektiv-idealen Bewußtseinserscheinungen, für deren Gebiet die physikalischen Gesetze jedenfalls nicht gelten, in welchen vielmehr andersartige psych. Gesetze an ihre Stelle treten. Nichthypothetisch sind unmittelbar die Gesetze der Phoronomie od. reine Bewegungslehre, die ebenso gewiß, aber auch ebenso rein formal u. realitätslos sind wie die der Math. u. Logik. Alles was die Formeln der Mechanik u. Energetik von denen der Phoronomie od. reinen Bewegungslehre unterscheidet, ist rein hypothetischer Natur, Gleichviel worin man den Unterschied suchen mag. Das bewegliche Rad in der Natur ist uns niemals durch Erfahrung gegeben, sondern erst das Denken zwingt uns die Hypothese auf, daß unseren Erfahrungen so etwas als Bedingung ihrer Verständlichkeit u. Gesetzlichkeit vorausgesetzt werden muß. 213

Nun unter diesen realistischen Voraussetzungen (Transcendentale Gültigkeit von Kausalität, Gesetzlichkeit u. dreidimensionalem, dem Raumsprechen

Beziehungsgegeben od. mitteilbaren Räumlichkeit), also nur unter Gleichsetzung der jenigen „Natur", von deren Gesetze die Physik redet, mit einem kausale, zeitliche u. mindestens raumähnlichen (topoïden) Reiche der Dinge an sich, kann vom Naturgesetze im Unterschied vom psych. Gesetzen die Rede sein. Nur wenn die Naturgesetze sich in einer von unserem Denken unabhängigen Gebiete abspiele, nur dann kann sie zur Erklärung dafür dienen, dass die denknotwendige Folge unserer Bilder stets wieder die Bilder sind von naturnotwendiger Folge des Unbekannten, das sie unseren Sen. abbilden od. symbolisiere. 215

Die Aufgabe einer künftigen Physik wird es sein, das Prinzip der quantitativen Exaktheit auch auf die Berechnung der Wahrscheinlichkeits-Koëfficienten der Gesetzes-, u. Ursachen-induktionen zu übertragen u. zugleich bei jeder Hypothese zu bestimmen, wie gross der wahrscheinliche Fehler des für sie berechneten Wahrscheinlichkeits-Koëfficienten ist. 218–9

Die Kausalität liegt ausserhalb des Reiches der Mathematik, die nicht einmal das Werden u. Wachsen einer variabelen Funktion, sondern immer nur ihre Grössenwert an einer bestimmten, aus der Thatsache der Veränderung herausgerissenen gedanklich fixierten Stelle bestimmen kann. Das reale Exkurs bewahrt auf exklärendem Denken hinaus-kommt. 221

Die Hypothesophobie ist eine ebensolche Kinderkrankheit der Physik, wie der Glaube an absolute Gewissheit ihrer Lehren. 222

Dieser Standpt (Exkl. Immanenz od. immanente Ph.) (Mach, Helm) beruht auf der Nichtunterscheidung des naiven Realismus v. absoluten

Illusionismus u. der daraus hervorgehenden Identifikation beide.
Seine Unterscheidbarkeit wird dadurch praktisch unschädlich, dass er bloss in rein theoretischen Erwägungen seine illusionistische Seite hervorkehrt, in physikalischer Betrachtung aber sich ganz naiv-realistisch gebärdet, d. h. er verhält sich in letzterer ganz ebenso wie die ganze Physik, die auch vor aller Erkenntniskritik unbekümmert geblieben ist, nur dass er sich streng auf die Gesetze der Veränderung u. Bewegung beschränkt u. das Veränderliche od. Bewegliche im ergostatischen Sinne bearbeitet. 216

Helmholtz, Zählen und Messen, erk. th.
betrachtet. H. — Wissensch. Abhandl.
III.

Was ist der obj. Sinn davon, dass
wir Verhältnisse reeller Objekte durch
benannte Zahlen als Grösse aus-
drücken, u. mittst welchen Bedingungen
(359) können wir dies thun? Diese
Frage löst sich, wenn wir finden
werden, in zwei einfachere auf, nämlich:
1. Was ist der obj. Sinn davon,
dass wir zwei Objekte in gewisser Be-
ziehung für gleich erklären?
2. Welchen Charakter muss die
physische Verknüpfung zweier Objekte
haben, damit wir verschiedenen
Attribute derselben als additiv
verbunden, und diese Attribute
demzufolge als Grössen, die durch
benannte Zahlen ausgedrückt
werden können, ansehen dürfen?
Benannte Zahlen nämlich betrachtet
wir aus ihren Teilen, bezüglich
Einheiten, durch Addition zusam-
gesetzt.

360.

--- ~~Diese Reihenfolge ist in der Tat eine von Menschen, unseren Voreltern, die die Sprache ausgearbeitet haben, gegebene Normen od. Gesetz.~~
Mit der Reihenfolge der Z-zeichen hat dies nichts zu tun; wie die Zeichen in den verschiedenen Sprachen verschieden sind so könnte auch ihre Reihenfolge willkürlich bestimmt werden, wenn nur unabänderlich irgend eine bestimmte Reihenfolge als die normale od. gesetzmässige festgehalten wird. Diese Reihenfolge ist in der Tat eine von Menschen, unseren Voreltern, die die Sprache ausgearbeitet haben gegebene Norm od. Gesetz. Ich betone diesen Unterschied, weil das angebliche „Natürliche" der Z-reihe mit der unvollständigeren Analyse des Begriffs der Z. zusammenhängt. Die Mathematiker bezeichnen diese gesetzmässige Z-reihe als die der posit. ganz. Zahlen 360

~~Das Zählen ist ein Verfahren, welches darauf beruht, dass wir uns im Stande finden, die Reihenfolge, in der Bewusstseinsacte zeitlich nach einander eingetreten sind, im Gedächtnis zu behalten. 360~~

~~Die Zahlen dürfen wir zunächst als eine Reihe willkürlich gewählter Zeichen betrachten, für welche nur eine bestimmte Art des Aufeinanderfolgens als die gesetzmässige, od. nach gewöhnlicher Ausdrucksweise „natürliche" von uns festgehalten wird. Die Bezeichnung der „natürlichen" Zahlenreihe hat sich wohl nur an eine bestimmte Anwendung des Zählens geknüpft, nämlich an die Ermittlung der Anzahl gegebener reeller Dinge. 360.~~

Die Zahlenreihe ist unserem Gedächtnis ausserordentlich viel fester eingeprägt als jede andere Reihe, was unzweifelhaft auf ihrer viel häufigeren Wiederholung beruht. Wir brauchen sie deshalb auch vorzugsweise, um durch Anknüpfung an sie die Erinnerung an die Reihenfolge in unserem Gedächtnis zu festigen, d. h. wir brauchen die Zahlen als Ordnungszahlen 361

"In dem wir von diesem eins nach dem andern den gezählten Haufen zuwerfen, folgen die Zahlen bei einem natürl. Vorgang auf einander in ihrer gesetzmässigen Reihe." "Nach den vorausgegangenen Erörterungen ist jede Z. nur durch ihre Stellung in der gesetzmässigen Reihe bestimmt. Das Zeichen Eins legen wir demjenigen Glied der Reihenfolge bei, mit dem wir beginnen. Zwei ist die Zahl, welche unmittelbar, d. h. ohne Zwischenschiebung einer andern Z., in der gesetzmässigen Reihe auf Eins folgt. 361 Drei ist die Z., die ebenso unmittelbar auf zwei folgt u.s.w. Ein Grund, diese Reihe irgendwo abzubrechen, oder in ihr zu einem schon früher gebrauchten Zeichen zurückzukehren, ist nicht vorhanden. 362 (Zeile 2. -7)

~~Das Zeichen Eins legen wir demjenigen Gliede der Reihenfolge bei, mit dem wir beginnen. 361~~

Wir betrachten nunmehr die normale Zahlenreihe als festgestellt und gegeben. Jetzt können wir ihre Glieder selbst als eine in unserem Bew. gegebene Reihe von Vorstellungen betrachten, deren Ordnung von einem beliebig gewählten Glied ab wir wieder durch die von Eins beginnende normale Zahlenreihe bezeichnen können. 363

Hiermit ist die Reihe der für Begründung der Arithmetik notwendige

Wenn ich die vollständige Zahlenreihe von 1 bis n brauche, wo jedem Elemente der Gruppe eine Z. zugeordnet, so nenne ich n die Anzahl der Glieder d. Gruppe. 371. Zeller, S. 32

Axiome der Addition sind dem nur aus innerer Anschauung entnommenen Begriff der Zahlen n. der Summe, von den wir ausgegangen sind, es zu weisen, n. zugleich die Uebereinstimmung des Ergebnisses dieser Art der Addition mit den, welche aus dem Zählen von äussern zählbaren Objekten hergeleitet werden kann 373.

Solche Objekte, die in irgend einer bestimmten Beziehung gleich sind n. gezählt werden, nennen wir die Einheit d. Zählung, die Anzahl derselben bezeichnen wir als eine benannte Z., die besondere Art der Einheit, die sie zusammenfasst, die Benennung der Zahl 374.

Objekte od. Attribute von Objekten, die mit ähnlichen verglichen den Unterschied des grössern, gleich od. kleiner zulassen, nennen wir Grössen. Können wir die 374 durch eine benannte Z. ausdrücken, so nennen wir diese benannte Z. das Mass der Grösse, das Verfahren, wodurch wir die benannte Z. finden, Messung der Grösse. Uebrigens gelange wir in vielen tatsächlich ausgeführten Untersuchungen nur dazu, die Messung auf arithmetisch gemachte, od. durch das gebrauchte Instrument gegebene Einheit zurückzuführen, dann haben die Zahlen, die wir finden, nur den Wert von Verhältniszahlen, bis jene Einheit auf allgemein bekannte (absolute Einheiten der Physik) zurückgeführt sind 375.

Die grosse Vereinfachung u. Uebersichtlichkeit der Auffassung, die wir durch Rückführung der bunten Mannichfaltigkeit der uns vorliegenden Dinge u. Veränderungen auf quantitative

Verhältnisse erreichen, ist tief in Wesen unserer Begriffsbildung begründet. Wenn wir den Begriff einer Classe bilden, fassen wir in ihm alles zusammen, was bei den Objekten, die in die Classe gehören, gleich ist. Wenn wir ein physisches Verhältnis als benannte Zahl auffassen, haben wir aus dem Begriff ihres Einheiten auch alles entfernt, was ihnen als verschieden in der Wirklichkeit anhaftet. Sie sind Objekte, die wir nur noch als Exemplare ihrer Classe betrachten, und deren Wirklichkeit nach der unterschiedlichen Richtung hin auch nur davon abhängt, dass sie solche Exemplare sind. In den aus ihnen gebildeten Grössen bleibt dann nur der zufälligste

der Unterschiede, die die Anzahl stehen

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Axiome 1. Wenn zwei Grössen einer dritten gleich sind, sind sie auch einander gleich.

A. 2. Associationsgesetz der Addition nach H. Grassmanns Bezeichnung:

$$(a+b)+c = a+(b+c)$$

A. 3. Commutationsgesetz der Addition

$$a+b = b+a$$

A. 4. Gleiches zu Gleichen addirt gibt Gleiches.

A. 5. Ungleiches zu Ungleichen addirt gibt Ungleiches (257)

A. 6. Wenn zwei Zahlen verschieden sind, muss eine von ihnen höher sein als die andre. (267)

2, 3, ... ist eine Folge (Helmholtz)

$(a+b)+1 = 1+(a+b)$ Folge

$a+1 = 1+a$ Folge Math. Induction

An Essay on the Foundations of Geometry,

Bertrand Russel

Those who held — as was generally held on the Continent — that certain knowledge, independent of exp was possible about the real world, had only to point to Geometry: none but a madman, they said, would cast doubt on its validity, & none but a fool would deny its aby ~~validity~~ reference. The English Empiricists, in this matter, had, therefore, a somewhat difficult task; either they had to ignore the problem, or, like Hamilton & Mill, they ventured on the assault. They were driven into the apparently paradoxical assertion that Geom., at bottom, had no certainty of a different kind from that of Mechanics — only the perpetual presence of spatial impressions, they said, made our experience of the truth of the axioms so wide as to seem

absolute certainty 1

In metrical Geom, the axioms will fall into two classes: (1) Those common to Euclidean & non-Euc. spaces. These will be found, on the one hand, essential to the possibility of measurement in any continuum, & on the other hand, necessary properties of any form of extensility with more than one dimension. They will, therefore, be declared a priori. (2) Those axioms which distinguish Euclid from non-Euc. spaces. These will be regarded as wholly empirical. The axiom that the no. of dimensions is three, however, though empirical, will be declared, since small errors are here impossible, exactly & certainly true of our actual world; while the two remaining axioms, which determine the value of the space-constant, will be regarded as only approximately known, & certain only within the errors of observation. 6

Everything in physical science, from the law of gravitation to the building of bridges, from the spectroscope to the art of navigation, would be profoundly modified by any considerable inaccuracy in the hypothesis that our actual space is Euclidean. The observed truth of phy. science, therefore, constitutes overwhelming empirical evidence that this hypothesis is very approximately correct, even if not rigidly true. 6 footnote

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While their (Riemann & Helmholtz) philosophical purpose was, to prove that all the axioms of Geom. are empirical, & that a different content of our exp. might have changed them all, the unintended result of their mathematical work was, if I do not mistake, to afford

material for an a priori proof of certain axioms. These axioms, though they believed them to be indispensably necessary, were always introduced in their math. work, before laying the foundations of non-Euclid. systems. 14

It is a pity that Riemann, in accordance with the metrical ideas of his time, regarded space as primarily a magnitude, or assemblage of magnitudes, in which the main problem consists is assigning quantities to the different elements or points, without regard to the qualitative nature of the quantities assigned. 15

But by the assumption, from the start, that space can be regarded as a quantity, he has been led to state the problem as: What sort of magnitude is space?

Riemann — 12½ pp. 3½ pp. — magnitude — 23 — 6½ — 7½ pp. — 1 pp. This can occur, he says, in several ways, as the simplest of which he assumes that the lengths of lines are independent of their position. One would be glad to know what other ways are possible for my part, I am unable to imagine any other hypothesis on which magnitude would be independent of place. 16 Rather than: What must space must be in order that we may be able to regard it as a magnitude at all? He does not realise, as up a precise coord. system, it is logically impossible to which the coordinates represent spatial magnitudes, without the axiom of free mobility. this axiom space as a species of manifold, holds or surfaces only therefore, though for analytical when the purpose it defines, most satisfactorily manifold of constant curvature is the nature of spatial magnitudes, Have our ideas of the measure- ments of curv. will only be really which lies in the nature of space free from reference to as a system of relations, i.e. the third dimension, when we are entirely to the possibility of regarding deal with a surface of it as a system of magnitudes at constant curv. all. 16 of curv. — a point which 19 The main result of Riemann's Riemann entirely overlooks. dissertation (2 pp.) may be summed condition 24 lines up as follows. Assuming it possible of the axiom = 2½ pp. 1½ — to apply magnitude to space

Space = Gaussian 2½ — 7½ pp. 2 — ½ Gaussian 1½ pp. ½

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Homogeneity

i.e. to determine its elements & figures by means of algebr. quantities, it follows that space can be brought under the concept of a manifold, as a system of quantitatively determinable elements. Owing, however, to the peculiar nature of spatial measurement, the quant. determinat. of space demands that magnitudes shall be independent of place — in so far as this is not the case, our measurement will be necessarily inaccurate. If we now assume, as the quant. relat. of distance between two elements ~~the~~ the square root of a quadratic funct. of the coordinates — a formula subsequently proved by Helmholtz & Lie — then it follows, since magnitudes are to be independent of place, that space must, within the limits of observation, have a constant measure of

But the arbitrary & conventional nature of distance, as maintained by Poincaré & Klein, arises from the fact that the two fixed pts, required to determine our distance in the proj. sense, may be arbitrarily chosen, & although, when our choice is once made, any two pts have a definite distance, yet, according as we make that choice, dist. will become a different funct. of the two variable pts. The ambiguity thus introduced is unavoidable

curvature, or must, in other words, be homogeneous in all its parts. ---------- Riemann, in spite of this desire to prove that all the axioms can be dispensed with, has nevertheless, in his math. work, retained three fundamental axioms, namely Free Mobility, the finite integral no. of dimensions, & the axiom that two points have a unique rel. namely distance. 22.

To sum up: Quantities, as used ~~and~~ in proj. Geom., do not stand for spatial magnitudes, but are conventional symbols for purely qualit. spatial relations. But distance, quâ quantity, presupposes identity of quality, as the condition of quant. comparison. Distance in the ordinary sense is, in short, that quant. rel. between two points on a line, by which their difference from other points

on proj. principles; but are we to conclude, ~~that proj. G.~~ from this, either it is really unavoidable? Must we not rather conclude that proj. G. cannot adequately deal with dist.? ---- Before we can distinguish the two fixed pts, therefore, from which the proj. definition starts, we must already suppose some relat. between any two pts on our line, in which they are independent of other pts, & this rel. is dist. in the ordinary sense. ---- Dist. in the ordinary sense remains a rel. between two pts, not between four; & it is the failure to perceive that the proj. sense differs from & cannot 35/ supersede, the ordinary sense, which has given rise to the views of Klein & Poincaré. The question is not one of convention, but of the irreducible metrical properties of space.

can be defined. The proj. definition, however, being unable to distinguish a collection of less than four points from any other on the same straight line, makes distance depend on two other points, besides those whose rel. it defines. No name remains, therefore, for distance in the ordinary sense, & many proj. Geometers, having abolished the name, believe the thing to be abolished also, & are inclined to deny that two points have a unique rel. at all. 36

Now Cayley showed that in Euclid G., distance may be expressed as the limit of the logarithm of the anharmonic ratio of the two pts & the (coincident) points at infinity on their straight line, while, if we assumed that the points at infinity were distinct, we obtained the formula for distance in hyperbolic or spherical G, according as these pts were real or imaginary. Hence it follows that, with the proj. definit of distance, we shall obtain precisely the formulae of hyp., or parab or spherical G. according as we choose the point, to which the value ∞ is assigned, at a finite infinite or imaginar distance (in the ordinary sense) from the pt to which we assign the value 0. Our straight line remains, all the while, an ordinary Euclid straight line. But we have seen that the proj. definit of distance fits with the Euclid definit only when the two fixed pts to which it refers are suitably chosen. Now the ordinary means of distance is required in non-Euclid. as in Euclid G. Geometries — indeed, it is only in metrical properties that these Geometries differ. Hence our Euclid straight line, though it may serve to illustrate other Geometries than Euclid's, we are

To speak, as Klein does, of ordinary planes with hyp. or ell. measures of dist., is either to incur a contradiction, or to forego any metrical meaning of dist. Instead of ord. planes we have surfaces like Beltrami's, of const. measure of curvature; instead of Euclid's space, we have hyp. or spherical space. At the same time, it remains true that we can, by Klein's method, give a Euclidean meaning to many symbolic propositions in non-Eucl. G.

~~still in the domain of purely proj.~~ can only be dealt with correctly by Euclid. Where we give a different definition of distance from Euclid's, we are still in the domain of purely projective properties, & derive no information as to the metrical properties of our straight line. But the importance, to Metageom., of this new interpretation lies in the fact that, having independently established the metrical formulae of non-Eucl. spaces, we find, as in Beltrami's Saggio, that these spaces can be related, by a homographic correspondence, with the points of Euclidean space; & that this can be effected in such a manner as to give, for the distance between two points of our non-Euclid. space, the (37) hyperbolic or spherical measure of distance for the corresponding points of Euclidean space. 38.

For by substituting, for dist. the logarithm above alluded to, we obtain, from the non-Eucl. results which follows from the ordinary Euclid. axioms. This correspondence removes, once for all, the possibility of a lurking contradiction in metageometry, since, to a proposition in the one, corresponds one & only one prop. in the other, & contradicting results in one respt. therefore would correspond to contradicting results in the other. Hence metageom. cannot lead to contradictions, unless Euclidean G. at the same ~~time~~ moment, leads to corresponding contradictions. Thus the Euclidean plane with hyp. or ellipt. measure of dist, though either contradicting or not metrical as an independent notion, has, as a help in the interpretation of non-Euclidean results, a very high degree of utility (39).

The general definition of a group is as follows: If we have any no. of independent variables

p.40 Newcomb (Crelle J. vol. 83), ellipt. g. & spherical g. [illegible]

$x_1, x_2 \dots x_n$ & any series of transformations of these into new variables — the transformations being defined by equations of specified forms, with parameters varying from one transformation to another — then the series of transformations form a group if the successive application of any two is equivalent to a single member of the original series of transformations. The group is continuous, when we can pass, by infinitesimal gradations within the group, from any one of the transf. to any other. 47

Lie: motion mobility of body :: group of transformations :: [illegible] Geom. : [illegible]

At first, to discredit the Transcend. Aesthetik seemed, to Metageometers, as important as to advance their science; but from the works of Cayley, Klein or Lie, no reader could gather that Kant had ever lived. We have also seen, however, that as the interest in phil. waned, the interest for phil. increased: as math. results shook themselves free from phil.

Axioms of metrical G. 1. Free mobility. It finds the analyt. expression of these condition in the existence of a space-constant, or constant measure of curvature, which is equivalent to the homogeneity of space. 2. space has a finite integral no. of dimensions. (finite no of coord.) 3. Distance. There exists one rel. betw. any two pts, which can be preserved unaltered in a combined motion of both points, & which, in any motion of a system as one rigid body, is always unaltered. (51)

Helmholtz : [illegible] Axiom of monodromy [illegible] If (n-1) pts of a body remain fixed, so that every other pt can only describe a certain curve, then that curve is closed. (28) [illegible] Lie, [illegible] d. Grundl. d. Geom. Leipziger Berichte 1890)

controversies, they assumed gradually a stable form, from which further development, we may reasonably hope, will take the form of growth, rather than transformation. — — When this stable stage has been attained, it is time for phil. to borrow of Science, accepting its final premises as those imposed by a real necessity of fact or logic. 50

But metrical Geom. though it is historically prior, is logically subsequent to proj. G. For proj. G. deals directly with that qualit. likeness which the judgement of quant. comparison requires as its basis. 51

The Axioms of proj. G., in fact, may be roughly stated thus:—

1. Space is continuous & infinitely divisible; the zero of extension, resulting from infinite division, is called a Point. All points are qualitatively similar, & distinguished by the mere fact that they lie outside one another.

2. Any two points determine a unique figure, the straight line; two straight lines, like two points, are qualitatively similar, & distinguished by the mere fact that they are mutually external.

3. Three points not in one straight line determine a unique figure, the plane, & four points not in one plane determine a figure of three dimensions. This process may, so far as can be seen à priori, be continued, without in any way interfering with the possibility of proj. G., to five or to n points. But proj. G. requires, as an axiom, that the process should stop with some positive integral no. of points, after which any fresh point is contained in the figure determined by those already given. If the process stops with (n+1) points, our space is said to have n dimensions. 52

[illegible] — Helmholtz [illegible]

correspond [illegible] quant [illegible]

Russel [illegible] Kant [illegible] Geom. [illegible]
apodeictic [illegible] a priori → subject.
[illegible] subj. & a priori [illegible]
Geom. is apodeictic [illegible] Metageom.
[illegible] Euclid [illegible] apod.
certainty [illegible]
[illegible]

The actual space we know, it may say, is admittedly Euclidean, & is proved, without any reference to Geom. to be a priori; hence Euclid has apodeict. certainty, & non-Euclid stands ~~and~~ condemned. To this it is no answer to urge, with the metageometers, that non-Euclidean systems are logically self-consistent; for Kant is careful to argue that geom. reasoning, by virtue of our intuition of space, is synthetic, & cannot, though a priori, be upheld by the principle of contradiction alone. (K. d. r. V. II ed. p. 39 "So werden auch alle geom. Grundsätze -- abgeleitet.) Unless non-Euclideans can prove, what they have cer-

tainly failed to prove up to the present, that we can frame an intuition of non-Eucl. space, 56 | Kant's position cannot be upset by Metageom. alone, but must also be attacked, if it is to be successfully attacked, on its purely phil. side. 57

It will be my aim to prove that the distinct. of synth. & analyt. judgements is untenable, & further (1) that the principle of contrad. can only give fruitful results on the assumpt. that experience in general, or, in a particular science, some special branch of exp. is to be formally possible; (2) that the first two arguments of the Trans. Aesth. suffice to prove, not Euclid. space, but some form of externality — which may be sensational or intuitional, but not merely conceptual — a necessary prerequisite of exp. of an external world. 57

I shall contend, as a result of these conclusions, that those axioms, which Euclid & Metageom. have in common,

So much, then, would appear to result from Kant's argument that experience of diverse but interrelated things demands, as a necessary prerequisite, some sensational or intuitional element, or percept, by which we are led to attribute complexity to objects of percept; that this element, in its isolation may be called a form of externality; & that those properties of this form, if any such be found, which can be deduced from its mere function of rendering experience of interrelated diversity possible, are to be regarded as a priori.

62. coincide with some properties of any form of externality which are deducible, by the pr. of contrad., from the possibility of experience of an external world. These properties, then, may be said, though not quite in the Kantian sense, to be a priori properties of space, & as to these, I think, a modified Kantian position may be maintained. But the question of the subj. or obj. nature of space may be left wholly out of account during the course of this discussion, which will gain by dealing exclusively with logical, as opposed to psychological points of view. 57.

Herbart, [illegible] (1 section of Psychologie) [illegible] But his psych. theory of space, his construction of extension out of series of points, his comparison of space with the tone & colour-series, his general preference(62) for the discrete above the continuous, & finally his belief in the great importance of classifying space with other forms of series (Reihenformen), gave rise to many of Riemann's epoch-making speculations, & encouraged the attempt to explain the nature of space by its analyt. & quant. aspect alone. (Compare, Erdmann, p. 30) 63

(Rieman [illegible]) It was thus assumed, to begin with, that spatial figures could be regarded as magnitudes, & the axioms which emerged, accordingly, determined only the particular place of these among the many algebraically possible varieties of magnitudes. The resulting formulat. of the axioms — while, from the math. standpt. of metrical Geom., it was almost wholly laudable — must, from the standpt. of phil. be regarded, in my opinion, as a petitio principii. For when we have arrived at regarding spatial figures as magnitudes, we have already traversed the most difficult part of the ground. The axioms

of metrical Geom — & it is metrical Geom. exclusively, which is considered in Riemann's Essay — will appear to be divisible into two classes. Of these, the first class — which contains the axioms common to Euclid & the Lobatchevskian Geom., the only axioms seriously discussed by Riemann — are not the results of measurement, nor of any conception of magnitude, but are conditions to be fulfilled before measurement becomes possible. The second class only — those which express the difference between Euclidean & non-Euclidean spaces — can be deduced as results of measurement or of conceptions of magnitude. As regards the first class, on the contrary, we shall see that the relativity of position — by which space is distinguished from all other known manifolds, except time — leads logically to the necessity of three of the most distinctive axioms of Geom., & yet this relativity cannot be called

a deduction from conception of magnitude. In analyt. Geom. — owing to the fact that coordinate systems start from points, & hence build up lines & surfaces, it is easy to suppose that points can be given independently of lines & of each other, & thus the relativity of position is lost sight of. The error thus suggested by mathematics was probably reinforced by Herbart's theory of space, which, by its quasi sensual character, appeared to him to facilitate a construction out of successive points, & to which Riemann acknowledges his indebtedness both in his Dissertation & elsewhere. The same error reappears in Helmholtz, in whom it is probably due wholly to the methods of analyt. Geom. It is a striking fact that, throughout the writings of these two men, there is not, so far as I know, one allusion to the relativity of space-position, that property of space from which the richest quarry of

of consequences can be extracted. This is not a result of any concept of magnitude, but follows from the nature of our space intuition; yet no one, surely, could call it empirical since it is bound up in the very possibility of locating things there as opposed to here.

64.

Indeed we can see, from a purely logical consideration of the judgment of quantity, that Riemann's manner of approach to the problem can never, by legitimate methods attain to a philosophically sound formulation of the axioms. For quantity is a result of comparison of two qualitatively similar objects, & the judgment of quantity neglects altogether the qualitative aspect of the objects compared. Hence a knowledge of the essential properties of space can never be obtained from judgments of quantity, which neglect these properties, while they yet presuppose them. As well might one hope to learn the nature of man from a census.

64.

We must entirely dissent, therefore, from the disjunct which underlies Riemann's phil. of space. Either the axioms must be consequences of general conceptions of magnitude, he thinks, or else they can only be proved by experience. Whatever can be derived from general conceptions of magnitude, we may retort, cannot be an a priori objective of space: for all the necessary objectives of space are presupposed in any judgment of spatial quantity, & cannot, therefore, be consequences of such a judgment. Riemann's disjunct, accordingly, since one of its alternatives is obviously impossible, really begs the question.

65.

The conditions of measurement

We do not require superposit. to measure quantities corresponding to different tones or colours; these can be discovered by analysis of single tones or colours.

themselves, though not results of any concept of magnitude, will be a priori, if it can be shown that, without them, experience of externality would be impossible. 65.

Do these magnitudes measure the elements of the manifold, or the relations between elements? This is surely a very fundamental point, but it is one which Riemann never touches on. In the former case, the superposition which he speaks of becomes unnecessary, since the magnitude is inherent in the element considered. — tone, colour in mag. [illegible]

With space, on the other hand, if we seek for elements, we can find none except points, & no analysis of a point will find magnitudes inherent in it — such magnitudes are a fiction of Coord. Geom. 66.

What we have, in colours, is three fundamental qualitatively distinct elements,

[illegible] element [illegible] pt [illegible] rel. [illegible] superposit. [illegible] superposit. [illegible] free mobility [illegible] (67) [illegible] Riemann, manifoldness [illegible] pt rel. [illegible] metrical space [illegible] (67) His phil. is chiefly vitiated, to my mind, by this fallacy, & by the uncritical assumption that a metrical coordinate system can be set up independently of any axioms as to space-measurements. (69) ..... Thus he says (p. 267): "Now the emp. conceptions, on which spatial measurements are based, the concept of the rigid body & of the light-ray appear to lose their validity in the infinitesimal; it is therefore quite conceivable that the rel[s] of spatial magnitudes in the infinitesimal do not correspond to the presuppositions of G. & this would in fact have to be assumed, as soon as it would enable us to explain the phenomena more simply": from this conclusion I must entirely dissent. In very large spaces there might be a departure from Euclid, for they depend upon the axiom of parallels which is not contained in the axiom of free mobility; but in the infinitesimal, departure from Euclid could only be due to the absence of free mobility, which is once for all impossible.

axiom · allgem. Grössenbegriff is apriori: follows [illegible] empir. [illegible]

out of certain proportions of which we can build up all the other elements of the manifold — each of the resulting elements having the same combination of qualit. diversity & similarity as the three original elements. But in space, what could we make of such a procedure? 68.

Metaphysically, space has no elements, being mere relations between non-spatial elements. Mathematically, this fact exhibits itself in the self-contradictory notion of the point, or zero magnitude in space, as the limit of our search for spatial elements. 68.

(Helmholtz: a priori [illegible] 184)

Helmholtz's criterion of apriority is difficult to discover, as he never, to my knowledge, gives a precise statement of it. From his discussion of physical & transcendental Geom. however, it would appear that he regards as empirical whatever

The main topics, for a criticism of Helmholtz, are three: First, his criterion of the a priori; second, his discussion with Land as to the imaginability of non-Eucl. spaces; third — & this is by far the most important of the three — his theory of the dependence of Geom. on Mechanics. 71.

70.

applies to emp. matter: for he there maintains, that even if space were a priori form, yet any Geom. which aimed at an application to Physics, would, since the actual places of bodies are not known a priori, be necessarily empirical. It seems the more probable that he regards this as a possible criterion, as it is adopted, in several passages, by his disciple Erdmann, & as strange a task could hardly be accepted by a philosopher, unless he had found it in his master. I have called this a strange task, because it seems to me completely to ignore the work of the Crit. Phil. for if there is one thing which, one might have hoped, had been made sufficiently clear by Kant's Critique, it is this, that knowledge which is a priori, being itself the condition of possible experience, applies to — & in Kant's view, applies only — empo

He says, for ex., that if our exp. showed us only bodies which changed their shapes in not — we should not arrive at the axiom of Congruence, which he presumes accordingly to be empir. But I shall endeavor to prove that without the axiom of Cong. experience of spatial magnitude would be impossible. If my proof be correct, it follows that no exp. can ever suggest spatial magnitudes which contradict this axiom — a possibility which Helmholtz admits. Helmholtz & Erdmann, therefore,

nowhere discuss, in setting up this test, without discussion, simply ignore the existence of Kant & his supp. the possibility of a transcend. argument.

71

exp. thus this second British thought perfectly sound, Ra- tional, is also to be found in Helmholtz, & seems always an accompa- nying to transcendental argument, might, by a different exp. have been rendered different — so this criterion a to the condi- tions of possible contends — must itself be dependent experience. on exp, & so emp. This criterion seems But this perfectly sound, but Helm's use of accompa- niment is it is usually vitiated by his neglecting seldom to to prove the possibility of the different- be found exp in quest.

in Helmholtz.

72

72.

Wenn die Reihe der Sinneseindrücke vollständig u. eindeutig angegeben werden kann, muss man m. E. die Sache für ausschaulich tot vorstellbar erklären. (Helm. Vorträge u. Reden II. 234)

This makes clear, what also appears from his manner of proof, that he regards things as imaginable

which can be described in conventional terms. 73.

Non-Eucl. space, conceivable not imaginable. 73-74.

(1) Congruence may be taken to mean — as Helmholtz would certainly seem to desire — that we find actual bodies, in our mechanical experience, to preserve their shapes with approximate constancy, & that we infer, from this experience, the homogeneity of space. This view, in my opinion, radically misconceives the nature of measurement & of the axioms involved in it. For what is meant by the non-rigidity of a body? We mean, simply, that it has changed its shape. But this involves the possibility of comparison with its former shape, in other words, of measurement. In order, therefore, that there may be any question of rigidity or non-rigidity, the measurement of spatial magnitudes must be already possible. It follows that measurement cannot, without a vicious circle, be itself derived from exp. of rigid bodies. Geom. measurement, in fact, is the comparison of spatial magnitudes, & such comparison involves the homogeneity of space. That is, therefore, a logical prerequisite of all exp. of rigid bodies, & cannot be the result of such experience. Without the homogeneity of space, the very not of rigidity or non-rigidity could not exist, since these mean, respectively, the constancy or inconstancy of spatial magnitudes in pieces of matter, & both alike, therefore, presuppose the possibility of spatial measurement.

76.

Congruence asserts, in short, that a body can, so far as mere space is concerned, move without change of shape; rigidity asserts that it actually does so move — a very different proposition, involving obviously, as its logical prius,

the former geometrical proposition.

↓ 76.

This argument may be summed up by the following disjunct.: If bodies change their shapes in motion — & to some extent, since no body is perfectly rigid they must all do so — then one of two cases must occur. Either the changes of shape, as bodies move from place to place follow no geom. law, are not, for instance, functions of the amount or direction of motion; in which case the law of Causat. requires that they should not be effects of the change of place, but of some simultaneous non-geom. change, such as temperature. Or the changes are regular, & the shape S becomes, in a new position p, Sf(p). In this case, the law of concomitant variat. leads us to attribute the change of shape to the mere posit. motion, & shape thus becomes a funct. of

absolute posit.: But this is absurd, for position means merely a rel. or set of relations; it is impossible, therefore, that mere posit. should be able to effect changes in a body. Posit. is one term in a rel, not a thing per se; it cannot, therefore, act on a thing, nor exist by itself apart from the other terms of the rel. Thus Helmholtz's view, that Congruence depends on the exist. of rigid bodies, must, since it involves absolute posit., be condemned as a log. fallacy. Congruence is in fact an a priori deduct. from the relativity of posit. 76

(2). To use a scholastic distinct. we may say that matter is the causa essendi of space, but Geom. is the c. cognoscendi of Physics. Without a Geom. independent of Physics, Physics itself, which necessarily assumes the results of G., could never arise; but when G. is used in Physics, it loses some of its a priori certainty, & acquires the emp. & approximate

The axiom of free mobility, it may be said, is logically distinguishable from the assert. of rigid bodies, & may even be not emp.; but it is barren, even for pure G., without the aid of measures, which must themselves be emp. rigid bodies. (6
[illegible])

character which belongs to all accounts of actual phenomena. 78.

(3). The reference to matter is necessitated by the homogeneity of empty space. For so long as we leave matter out of account, one point is perfectly distinguishable from another, & a science of the relations of positions is impossible. 77.

Some sort of matter is essential to Geom. But this geom. matter is a more abstract & wholly different matter from that of Dynamics. In order to study space by itself, we reduce the properties of matter to a bare minimum: we avoid entirely the category of causality, so essential to Dynamics, & retain nothing, in our matter, but its spatial adjectives. 79.

Geom. requires, if it is to be practically possible, some body or bodies which are either rigid (in the dynamical sense), or known to undergo some definite changes of shape according to some definite law. — One or more such bodies are necessary to applied Geom — but only in the sense in which rulers & compasses are necessary. — The emp. element in practice, arising from the purely emp. nature of physical rigidity, means only that a shape is comparable to the emp. inaccuracies arising from the failure to find straight lines or circles in the world — which no one but Mill has regarded as rendering G. itself emp. or inaccurate. 81.

Erdmann, 87.

Erd. agrees throughout with the conclusions of Riem. & Helm., except on a few points of minor importance; & his views, as this agreement would lead one to expect, are neither emp. Indeed his logic seems — though I say this with hesitation — to be incompatible with any style but that of Mill: there is apparently

no distinction, to him, between the general & the universal, & consequently no concept not embodied in a series of instances. Such a theory of logic, to my mind, vitiates most of his work, as it vitiated Riemann's phil. (On the influence of Mill cf. Stallo, Concepts of Modern Physics, p. 216) 82.

abstract general / universal & concrete space [illegible]

Let us reflect on the theories involved in the above account of the conception of space as a magnitude. In the first place, it is assumed that conception cannot be formed unless we have a series of separate objects from which to abstract a common property — in other words, that the universal is always the general. In the second place, it is assumed that all definition is classification under a genus. In the third place, the conception of ~~a~~ magnitude, if I am not mistaken, is fundamentally misunderstood when it is supposed applicable to space as a whole. But in the fourth place, even if such a conception existed it could give none of the essential properties of space. 83.

Cf. Bergson.

1, 2 [illegible] universal [illegible] general [illegible] 3 [illegible] magnitude [illegible] judgement [illegible] comparison [illegible] quality [illegible] comparison [illegible] pure no. [illegible] pure quantity in Calculus [illegible] ~~magnitude [illegible] judgement~~ quality [illegible] space [illegible] quality [illegible] different spaces [illegible] qualitative difference [illegible] ~~space as a whole [illegible] judgement of magnitude~~ preclude [illegible] space [illegible] constant [illegible] quantitative comparison [illegible] qualitatively similar unit [illegible] space [illegible] space as a whole [illegible] judgement of magnitude [illegible] Again to proceed to a more phil. argument, two different spaces cannot coexist in the same world: we may be unable to decide between the alternatives of the disjunction, but they remain, none the less, absolutely incompatible alternatives. Hence we cannot get that coexistence of two spaces which is essential to comparison. The fact seems to be that Eri[illegible], in his admiration for R. & H. has fallen in with their math. bias, & assumed, as 85/ mathematicians

naturally tend to assume, that quant[ity]
everywhere & always applicable
& adequate, & can deal with more
than the mere comparison of things
whose qualities are already known
to be as similar ... 86.

[illegible] 4, [illegible] similar qualitative [illegible] quant.
[illegible] space
essential property [illegible] touch [illegible]

G, as Grassmann insists (Ausdehnungslehre
1844 2nd ed. pp. XXII, XXIII), ought not to be
classed with pure mathematics, for it
deals with a matter which is given
to the intellect, not created by it.
The axioms give the means by which
this matter is made amenable to
quantity, & cannot, therefore, be
themselves deduced from purely quant.
considerations ... 91.

G. & pure math.

Erdmann arith. axiom [illegible] emp. [illegible]
(165) G. [illegible] induct. science (172)

Lotze, R. [illegible] Metaphysik, [illegible]
[illegible] Metageom. [illegible] Eucl. [illegible]
[illegible]

Now the conditions to which a form
of externality must conform, in order
to be affirmed, are: 1st, of course that
it should be experienced, or legitimately
inferred from something experienced; but
secondly, that it should conform to
certain log. conditions, which may
be summed up in the relativity
of position. Now what Metag. has
done, in any case, is to suggest
the proof that the 2nd of these
conditions is fulfilled by non-Eucl.
spaces. Euclid is affirmed, therefore,
on the ground of immediate experience
alone, & both alike, as immediate
by log. necessity, is merely assertorical,
or if we prefer it, empirical. This
is the most import. sense, it seems
to me, in which non-Eucl. spaces
are possible. They are, in short,

a step in a phil. argument, rather than in the investigat. of fact: they throw light on the nature of the grounds for Euclid, rather than on the actual conformat. of space. Lotze .. de metag. .. im port [illegible]

If every conceivable behavior of light-rays can be explained, with Euclid, by physical causes, it must also be possible, by a suitable choice of hypoth. phys. causes, to explain the actual phenomena as belonging to a non-Eucl. space. Such a hypothesis would be rightly rejected by Science, for the present, on account of its unnecessary complexity. Nevertheless it would remain, for phil., a possibility to be reckoned with, & the choice could only be decided upon emp. grounds of simplicity. 100

Lotze. Helm. .. Flatland, sphereland. [illegible] Helm. [illegible] Metageom, [illegible] 101–8

Lotze: "I cannot persuade myself that one could, without the elements of homog. space, or of even form or define the presentat. of heterog. spaces, or of such as had variable measures of curvature." As though such spaces were ever set up by non-Eucl. mathematics! Lotze .. [illegible] 108

Non-Euclid. G. .. absolute magnitude [illegible] .. mag. .. relative [illegible]. All we require is, that it shall be impossible, while the rest of space is unaltered unaffected, to alter the magnitude of any figure, as compared with other figures, while leaving the rel. internal magnitudes of its parts unchanged. This construct., which is possible in Euclid, is impossible in Metag. We have to discuss whether such an impossibility renders non-Eucl. spaces logically faulty. 110

I do not believe that this is the case. For an undue reificat.

?

Non-Eucl. figures [illegible]

Characteristic of Non-Eucl. G.

of space would only arise if we were no longer able to regard posit as wholly relative, & as geometrically definable only by departure from other positions. But the relativity of posit is preserved by all spaces of constant curvature — in all of these, position can only be defined, geometrically, by relation to fresh positions. 111

the whole of metageom.; in short, is a proof that the relativity of posit is compatible with absolute magnitude, in the only sense required by non-Euclid spaces. 112.

If this be so, it is that branch of pure math. which Hausdorff, in his Grundzüge of 1844, felt to be possible, & endeavoured, in a brilliant failure, to construct without any appeal to the space of intuition. (sketch)

Proj. G., in so far as it deals only with the properties common to all spaces, will be found, if I am not mistaken, to be wholly apriori, to take nothing from exp. & to have, like Brill., a creature of the pure intellect for its object. 113

In the 1st place, it is important to realise that 118/ when coordinates are used, in proj. G, they are not coordinates in the ordinary metrical sense, i.e. the numerical measures of certain spatial magnitudes. On the contrary, they are a set of numbers, arbitrarily but systematically assigned to different pts, like the nos. of houses in a street, serving only, from a plist standpt, as convenient designation for point which the investigator wishes to distinguish. 119.

The distinction betwn various pts, tho' is not a result, but a condition of the proj. G, coordinate system. The coordinate system is a wholly arbitrary, & merely convenient set of marks, which in no way touches the essence of proj. G. What we must beg — with, in this domain, is the possibility of distinguishing various points from one another. This may be designated, with Veronese, (p. 224.) as the postulate of G. 119

All geom. reasoning is, in the last resort, circular: if we start by assuming points, they can only be defined by the lines & planes which relate them; & if we start by assuming lines & planes, they can only be defined

Indefinability of point: relativity of space-elements.

dualist

(130 also) proj. property [illegible] anharmonic ratio [illegible] four pts. [illegible] two pts. [illegible]
(130–131)

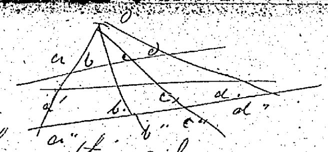
[illegible] any pair of pts. [illegible] projection [illegible] indistinguishable (120)
[illegible] any pair of lines [illegible] qualitatively indistinguishable [illegible] (121) + 130

by the points through which they pass. 120

The two mathematically fundamental things in proj. G. are anharmonic ratio, & the quadrilateral construction. Everything else follows mathematically from these two. [illegible] purely qualitative [illegible] Russell [illegible] 122.

I shall endeavour to show, in what follows, first, in what non-metrical ideas figures obtained from each other by proj. transform. (pr. & sect.) are qualitatively alike; secondly, what axioms, or adjectives of space, are involved in the precepts of proj. transf.; & thirdly, that these adjectives must belong to any form of externality with more than one dimension, & are, therefore, a priori properties of any possible space. 122.

Anharmonic ratio [illegible] non-metrical ideas [illegible] Two sets of four pts. each are defined as having the same anharmonic ratio, when (1) each set of four ~~lines~~ lies in one straight line, & (2) corresponding pts. of different sets lie two by two on four straight lines through a single pt. or when both sets have this relation to any ~~ot~~ third set. And reciprocally: Two sets of four straight lines are defined as having the same anharmonic ratio when (1) each set of four passes through a single pt., & (2) corresponding lines of different sets pass, two by two, through four points in one straight



line, or when both sets have this rel. to any third set.

Two sets of pts. or of lines, which have the same anharmonic ratio, are treated by proj. G. as equivalent: this qualit. equivalence replaces the quant. equality of metrical G., & is obviously included, by its definition, in the above account of proj. transformation in general. 123.

[illegible] project. & sect. [illegible] operation [illegible] proj. G. [illegible]

In other words, since what can really be taken, without contradiction, as the term of a spatial rel., is unextended, we must take, as the term to be used in G., where we cannot go outside space, the least spatial thing which G. can deal with, the thing which, though in space, Contains no space; & this thing ~~is~~ we define as the pt. (It is important to observe that this definition of the Pt. introduces metrical ideas. Without metrical ideas, we saw Points

nothing appears to give the Pt precedence of the straight line, or indeed to distinguish it conceptually from the straight line. A reference to quantity is therefore inevitable in defining the Pt if the definition is to be geometrical. A non-metrical def would have to be also non-genetic. 128.

Now quantitative comparison depends on a recognised identity of quad; the recognition of quad identity, therefore, is logically prior to quant. & presupposed by every judgement of quant now

We can prove, I think, that two figures, which are projectively related, are qualitatively similar. Let us — Immediate intuition can distinguish the two figures, but qual discrimination cannot do so. ~~Thus we obtain a proj transformation of four lines into~~ Thus the only reason, with proj G, for not regarding proj figures as

actually identical, is the intuitive percept of difference of position. This is fundamental, & must be accepted as a datum. It is presupposed in the distinct of various points, & forms the very life of G. This is in fact, the essence of the work of a form of (131) externality, which within forms the subject-matter of proj G. 132.

The axioms which have been assumed in the above analysis & which, it would seem, suffice to found proj G, may be roughly stated as follows:

I. We can distinguish different parts of space, but all parts are qualitatively similar, & are distinguished only by the immediate fact that they lie outside one another. (Externality (Cf. Bergson))

II. Space is continuous & infinitely divisible; the result of infinite division the zero of extension, is called a point. (Cf. Weyl. Archimedisches Axiom)

III. Any two Pts determine a unique figure, called a straight line; any three in general determine a unique (Kontinuitäts Axiom (notwendig für Weyl) aber nicht notwendig für Hilbert)

All these axioms are philosophically interdependent, & may therefore be enunciated in many ways. 132

figure, the plane. Any four determine a corresponding figure of three dimensions, & for aught that appears to the contrary, the same may be true of any no. of pts. But this process comes to an end, sooner or later, with some no. of pts which determine the whole of space. For if this were not the case, no number of relations of a point to a collection of given pts could ever determine its relation to fresh pts, & G would become impossible. 132.

& that the external relations,

For it is assumed that a figure can be completely defined by its internal rels, which constitute its position, though they suffice to distinguish it from other figures, in no way affect its internal properties, which are regarded as qualitatively identical with those of figures with quite different external relations. --- We require for our principle, in short, what may be called the mutual passivity & reciprocal independence of two parts or figures of space.

Now when we consider what is involved in such absolute qual. equivalence, we find at once, as its most obvious prerequisite, the perfect homogeneity of space. 133

The homog. of space & the relativity of position are presupposed in the qualitative spatial comparison with which proj. G. deals. ~~with The latter is also~~ --- But these properties belong of necessity to any form of externality, & are thus a priori properties of all possible

This passivity & this independence involve the homogeneity of space, or its equivalent the relativity of position. 133

In Grassmann's profound phil. introduction to his Ausdehnungslehre of 1844, he suggested that Geometry, though improperly regarded as pure, was really a branch of applied math., since it dealt with a subject matter not created, like no., by the intellect, but given to it, & therefore not wholly subject to its laws alone. But it must be possible — so he contended — to construct a branch of pure math., a science, that is, in which our object should be wholly a creation of the intellect, which should yet deal, as Geom. does, with extension — extension as conceived, not as empirically perceived in sensation or intuition. 134

spaces

134. The distinction between Eucl. & non-Eucl. Gs, so important in metrical investigations, disappears in proj. G. proper. This suggests that proj. G., though originally invented as the science of Eucl. space, & subsequently of non-Eucl. spaces also, deals really with a wider concept, a concept which includes both, & neglects the attributes in which they differ. This concept I shall speak of as a form of externality. 134

And Grassmann's distinction, I contend, is the more important for Epistemology & the one to be adopted in distinguishing the a priori from the empirical. For what is merely intuitional can change, without upsetting the laws of thought, without making knowledge formally impossible, but what is purely intellectual cannot change, unless the laws of thought should change, & all our knowledge simultaneously collapse. 135

pure intellect ... G. ... hypothetical (necessary conviction & certainty [illegible])

Grassmann ... experienced externality ... to exist

empirical material ... contextual import ... proj. G. axioms ... pure form of externality 135

[illegible] 135

Externality [illegible] In the 1st place, externality is an essentially relative concept — nothing can be external to itself. To be external is to be another, with some rel. to that thing. Hence when we abstract a form of externality from all material content, & study it in isolation,

positin will appear, of necessity, as purely rel — a posit can have no intrinsic quality, for our form consists of pure extensivity, & extensivity contains no shadow or trace of an intrinsic quality. Thus we obtain our fundamental postulate, the relativity of posit, or, as we may put it, the complete absence, on the part of our form, of any vestige of thinghood.

137.

If we abstract the concept of extensivity, & endeavour to deal with it per se, it is evident that we must obtain an object alike destitute of element & of totality. For we have abstracted from the diverse matter which filled our form, while any element, or any whole, would retain some of the qualities of a matter. Either an element or a whole, in fact, would have to be a thing-in-itself, & would thus contain something not pure extensivity. Hence arises

when we regard a quantity as divisible ad infinitum, i.e. as divisible into parts, we necessarily regard it as more than a mere relation or adjection, since no mere rel. or adjective can be divided. For quantitative treatment, therefore, spatial rels must be hypostatized. — It is thus solely from the hypostatizing of rels, which mathematically requires, that the views of straight lines & planes as compared with plane areas, & it is from this hypostatizing that the difficulties of mathematics, spring (139)

thus again, our form contains notion of the pt, in the search for elements, neither elements nor totality, & unbounded extension, with the contradictory but only endless relation of the search for a completed whole, the terminus of which relation being excluded by our abstraction from the matter which fills our form.

137.

The diversity of content, which was possible only with the form of extensivity, has been abstracted from, leaving with it but the bare possibility of diversity, the bare principle of differentiation, itself uniform & undifferentiated. For if diversity presupposes such a form, the form cannot, unless it were contained in a fresh form, be itself diverse or differentiated.

137

Positions, we have seen, are defined solely by their rels to other positions. But in order that such definition may be possible, affinities no of rels must suffice, so that definite nos are possible, is admissible. A posit must be

Principles of dimensions

definable, therefore, if knowledge of our form is to be possible at all by some finite integral [139] no. of relat[ion]s to other posit[ion]s. Every rel. thus necessary for defin[ition] we call a dimens[ion]. Hence we obt[ain] the proposit[ion]: Any form of externality must have a finite integral no. of dimensions. 140.

The function in constituting the possibility of exp. which we demand of such forms, could not be accomplished by a one-dimensional form alone. For in a one-dimensional form, the various contents may be arranged in a series, & cannot, without interpenetrat[ion], change the order of contents in the series. But interpenetration is impossible, since a form of externality is the mere expression of diversity among things from which it follows that things cannot occupy the same posit[ion] in a form, unless there is another form by which to differentiate them. For without externality, there is no diversity. Thus two bodies [141] may

Time & Space as Correlates

In a one-dimensional space, for example, only a single object which must appear as a pt, or two objects at most, one in front & one behind, could ever be perceived. Thus two or more dim. seem an essential condit[ion] of anything ~~which~~ worth calling an experience of interrelated things. 142.

occupy the same space, but only at different times; two things may exist simultaneously, but only at different ~~times~~ places. A form of one dimens[ion], therefore, could not, by itself, allow that change of the relat[ion]s of externality by which alone a varied world of interrelated things can be brought into consciousness. 142.

If two things are mutually external, they must, since they belong to the same world, have some rel. of externality; there is, therefore, a rel[ation] of externality between two things. But since our form is homogeneous, the same rel. of externality may subsist in other parts of the form, i.e. while the two things considered alter their rel. of externality to other things. The rel. of externality between two things is, therefore, independent of other things. Hence, when we return to the abstract language of the form, two posit[ion]s have a rel. determined by these two posit[ion]s alone, & independent of other positions. 144.

The two posit[ion]s: intrinsic rel. in straight line 1 + y 2 ...
3 posit[ion]s : ... plane 1 ... 146

I wish to point out that proj. G. is wholly a priori; that it deals with an obj. whose properties are logically deduced from its definit., not empirically discovered from data; that its diversity in [definit., as a[illegible], is founded on the possibility of experience] rel, or multiplicity in unity; & that our whole science, therefore, is logically implied in, & deducible from, the possibility of such exp. 146.

In metrical G. on the contrary, we shall find a very different result. Although the geom. ~~construct~~ conditions which render spatial measurement possible, will be found identical, except for slight differences in the form of statement, with the a priori axioms discussed above, (48) yet the actual measurement — which deals with actually given space, not the mere intellectual constructions we have been just discussing — gives results which can only be known empirically & approximately, & can be deduced by no necessity of thought. The Eucl.

& non-E. spaces give the various results which are a priori possible; the axioms peculiar to Euclid — which are not properly axioms, but emp. results of measurement — determine, within the error of observation, which of these a priori possibilities is realised in our actual space. 147

proj. G. is = science of space ? +: i.e. metrical = Physics: i.e.: [illegible]

Metrical G. is, then, a necessary part of the science of space, & a part not included in descript. G. Its a priori element, nevertheless, so far as this is spatial & not arith., is the same as the ~~[illegible]~~ postulate of proj. G., namely, the homogeneity of space, or its equivalent, the rel. of position. We can see, in fact, that the a priori element in both is likely to be the same. For the a priori in metrical G. will be whatever is presupposed in the possibility of spatial measurement, i.e. of quant. spatial element. But such comparison presupposes simply

Geometry? 174, 95 [illegible] metrical G. [illegible]

a known identity of quot, the determination of which is precisely the problem of proj. 4. Hence the conditions for the possibility of measurement, in as far as they are not arith, will be precisely the same as those for proj. 4.

Metrical 4, therefore, though distinct from proj 4 is not independent of it, but presupposes it, & arises from its combination with extraneous idea of quantity. Nevertheless the math form of the axioms, in metr. 4, is slightly different from that form in proj. 4. The homogeneity of space is replaced by its equivalent, the axiom of free mobility. The axiom of the straight line is replaced by the axiom of distance. Two pts determine a unique quant. dist, which is unaltered in any motion of the two pts as a single figure. This axiom, indeed, will be found to involve the axiom of the straight line — such a quant. could not exist unless the two pts determined a unique curve — but its math. form

spatial magnitudes can be moved from place to place without distortion, or, shapes do not in any way depend upon absolute position in space. (Axioms of Num. Syst. 100)

is changed. Another important change is the collapse of the B. of duality. quant. can be applied to the straight line, because it is divisible into similar parts, but cannot be applied to the indivisible pt. We thus obtain a reason, which was wanting in discuss 4. for referring pts, as spatial elements, to straight lines & planes. Finally, an entirely new idea is introduced with quant, namely, the idea of motion. Not that we study motion, & that any of our results have reference to motion, but that they cannot, though in proj. 4. they could, be obtained without at least an ideal motion of our figures through space.

149

equality of magnitudes → superposition → free mobility.

The denial of the axiom of free mobility involves absolute position, an act of mere space, per se, on this [illegible] 15/

Erdmann .. (S. 60) [illegible]

Standard position [illegible]

[illegible] ds [illegible] (dist. p: coordinate [illegible])

p [illegible] ds. f(p) [illegible] p [illegible]

[illegible] f(p) [illegible]

[illegible] absolute Relativism [illegible] 152–3

[illegible] Helm. Riemann [illegible]

[illegible] ds, ds [illegible] p [illegible] f(p) ds, f(p) ds₂

[illegible] alteration of shapes during motion

[illegible] relative [illegible]

[illegible] ds, [illegible] f(p) ds, ds₂ [illegible] f(p) ds₂

[illegible] absurd [illegible] 153–4

Dimension [illegible] a priori [illegible]

[illegible] empirical [illegible] 162

[illegible] law of Gravitation [illegible]

emp. knowledge [illegible] different order [illegible]

[illegible] error [illegible]

[illegible] It follows that the certainty of our whole axiom, that the no. of dimensions is three, is almost as great as that of the a priori element, since this element leaves to sense a definite disjunction of discrete possibilities. 164

two pts. [illegible] rel. [illegible] homogeneity ([illegible] pt [illegible])

[illegible] rel. [illegible] intrinsic rel. [illegible] distance (164)

[Left margin notes:] follows a definite known law [illegible] magnitude [illegible] measurement [illegible] comparison [illegible] ds, ds [illegible] There is no sense, therefore, in supposing the two mag. unequal when separated, nor in supposing, consequently, that they have changed their mag. in motion. This senselessness of our hypothesis is the logical ground of the math. indeterminateness as to the law of variation. Since then, there is no means of comparing two spatial figures, as regards magnitude, except empirically, the only logically possible axiom, if spatial mag. is to be self-consistent, is the axiom of Free Mobility in the form first given above. 154 [illegible] (157)

absolute [illegible] [illegible]

(When we speak of the length of a curve, we can give a meaning to our words only by supposing the curve divided into infinitesimal rectilinear arcs, whose sum gives the length of an equivalent straight line; thus unless we presuppose the straight line, we have no means of comparing the length of different curves, & therefore can never discover the applicability of our definition.) 167

[illegible] two points [illegible] pt [illegible]

passivity of space, mutual independence of its parts [illegible] 165–166

two points [illegible] define [illegible] Straight line [illegible] distance [illegible] measure [illegible]

[illegible] dist. [illegible] straight line [illegible] shortest dist. [illegible] the distance [illegible]) 167

(167) A straight line is not the shortest distance, but is simply the distance between two pts. 168

Two points [illegible] dist. [illegible] only one rel. [illegible] pt [illegible] qualit. [illegible] 166

I propose to prove, (1) that if distance, as a quantity completely determined by two pts, did not exist, spatial magnitude would not be measured, (2) that dist. can only be determined by two pts, if there is an actual curve in space determined by two pts, (3) that the exist. of such a curve can be deduced from the concept of a form of externality, & (4) that the applicat. of quant. to such a curve necessarily leads to a certain magnitude, namely distance, uniquely determined by any two pts which determine the curve. The conclusion will be, if these propositions can be successfully maintained, that

[Left margin:] circular arc [illegible] circle [illegible] 2 pts [illegible] 167

(F 37) the rest of space. This peculiarity presupposes a measure of linear magnitude in general, & cannot upset the apriority of the axiom of the straight line. But it destroys, for pts having the peculiar antipodal rel. to each other, the argument which proved that the rel. between two pts could not, since it was unchanged by moti, have reference to the rest of space. Thus it is intelligible that for such special pts, the axiom breaks down, & an infinite no. of straight lines are possible between pair of pts.

10-[illegible] "py" the axiom of dist is a priori in the same double sense as the ax. of Fr. Mob (i.e. it is presupposed in the possibility of measurement, & it is necessarily true of any possible form of externality. 164)

[illegible Japanese marginal note] intrinsic rel. 172

If pts are defined simply by relat to other points, i.e. if all posit is rel, every pt must have to every other pt one, & only one, rel independent of the rest of space. This rel is the dist between two pts. Now a rel between two pts can only be defined by a line joining them — say further, it may be contended 173

Then, but unless we had started with assuming the general validity of the axiom, we could never have reached a point in which antipodal pts could have been known 172 to be peculiar, or indeed a posit which would

Dist & the straight line as relations uniquely determined by two pts, are thus a priori necessary to metrical G. But further, they are properties which must belong to any form of externality. Since this necessity for G was deduced from homogeneity & the rel of posit, & since these are necessary properties of any form of ext, the same argument proves both conclusions. We thus obtain, as in the case of Fr. Mob, a double aprioriity: The ax of Dist & its implication the ax. of the Straight Line, are, on the one hand, presupposed in the 173 possibility of spatial magnitude & cannot,

that a rel. can only be a line joining them. Hence a unique rel. involves a unique line, i.e., a line determined by any two of its pts. Only in a space which admits of such a line is linear magnitude a logically possible concept. But when once we have established the possibility in general, of drawing such lines, & therefore of measuring linear magnitudes, we may find that a certain magnitude has a peculiar (rel) to the constitution of space. The straight line may turn out to be of finite length & in this case its length will give a certain peculiar magnitude, the space constant. Two antipodal pts, that is, pts which bisect the entire straight line, will then have a rel. of magnitude which though unaltered by moti, is rendered peculiar by a certain constant rel. to (F)

have enabled us to give any quantitative definit whatever of particular pts. 173

therefore, be contradicted by any exp resulting from the measurement of space; while they are consequences, on the other hand, of the necessary properties of any form of externality which is to render possible exp of an external world.

[illegible] measurement [illegible] 174

Philosophical consequences 174-5

1. What rel can a purely log. & deductive proof like that from the nature of a form of ext. bear to an experienced subject matter such as space? [illegible marginal note] 190: subjective, psych. probl. [illegible]

2 How can we remove the contradicti which arise out of the rel. infinity, infinite div. & unbounded ext. of space? 177-8

Russell [illegible marginal notes] 173

The concept of a form of ext is the general concept, containing under it every logically possible intuit which can fulfill the functi actually fulfilled by space. 179

The form of ext is necessary to exp. but is not, on that account, to be declared purely subj. Of course, necessity for

Space forms, for Kant, no part of the data of sense, but is added by a subj. intuit., which he regards as not only logically, but psychologically, prior to objects in space. (See, Vaihinger, II, p 86–7, 168–171) 180

[illegible] Russell [illegible] Caird (I, 287) [illegible]

exp. can only arise from the nature of the mind which experiences; but it does not follow that the necessary conditions could be fulfilled, unless the obj. World had certain properties. The ground of necessity, we may safely say, arises from the mind; but it by no means follows that the truth of what is necessary depends only on the constitution of the mind 179

Objektivität d. R.

Space seems to perform two functions: on the one hand, it reveals things, by the eccentric projection of sensations, as external to the self, while, on the other hand, it reveals simultaneously presented things as mutually external. These [181] two functions, though often treated as coordinate & almost equivalent, seem to be widely different 182

X

Knowledge involves a recognition of diversity in rel. or if we prefer it, of identity in difference (184) [illegible] a mere particular [illegible] complex [illegible] one dimensional form [illegible] of ext. [illegible] time [illegible] change [illegible] change [illegible] causality [illegible] diversity [illegible] space [illegible] 184–6

[illegible]

The Kantian argument which was correct in asserting that real diversity, in our actual world, could only be known by the help of space — was only mistaken, so far as its purely log. scope extends, in overlooking the possibility of other forms of ext. which could, if they existed, perform the task with equal efficiency. In so far as space differs, therefore, from these other conceptions of possible intuitional forms, it is a mere experienced fact, while in so far as its properties are those which all such forms must have, it is a priori necessary to the possibility of exp. 186

It follows, that to suppose two things in the same pt. of space & time, is still a log. contrad. not because we have constructed the data of sense

How we are to account for the fortunate realizat– of these requirements (axi– 1–2) — whether by a pre-established harmony, by Darwinian adaptat– to our environment, by the subjectivity of the necessary element in sense-percept–, or by a fundamental identity ~~unity~~ between ourselves & the rest of reality — is a further quest– ~~to~~ belonging rather to metaphysics than to our present line of argument. (187) 1–2. Russell, 3½

Cf. Ins't exp.

out of logic, but because logic is dependent, as regards its applicat–, on the nature of these data. This instance illustrates, what I am anxious to make plain, that my argument has not attempted to construct the living wealth of sense-percept– out of "bloodless categories", but only to point out that, unless sense-percept contained a certain element, these categories would be powerless to grapple with it. 187

(1) Points. The antinomy of the pt — which arises wherever a continuum is given, & elements have to be sought in it — is fundamental to Geom. It has been given, perhaps unintentionally, by Veronese as the first axiom, in the form: "There are different pts. All pts are identical." (cp. S. 226). 189

A pt must be spatial, otherwise it would not fulfill the funct– of a spatial element; but again it must contain no space, for any finite extens– is capable of finite analysis. Pts can never be given in intuit–, which has no concern with the infinitesimal: they are a purely conceptual construct–, arising out of the need of terms between which spatial relat–s can hold. If space be more than relativity, spatial rel^s must involve spatial relata; but no relata appear, until we have analyzed our spatial data down to nothing. 189

The contradictory not– of the pt, as a thing in space without spatial magnitude, is the only outcome of our search for spat. relata. This reductio ad absurdum surely suffices, by itself, to prove the essential relativity of space. 190

Thus G. is forced, since it wishes to regard space as independent, to hypostatise its abstract–s, & therefore to invent a self-contradictory not– as the spatial element. A similar ~~not–~~ absurdity appears, even more obviously, in the not– of a

whole of space. The antinomy may therefore be stated thus: Space must, if knowledge of it is to be possible, be mere relativity; but it must also, if independent knowledge of it, such as G. seeks, is to be possible, be something more than mere relativity: since it is divisible & has parts. But we saw that knowledge of a form of ext. must be logically independent of the particular matter filling the form. How are we to extricate ourselves from this dilemma?

The only way, I think, is not to make G. dependent on Physics, but to give every geom. proposition a certain reference to matter in general. And at this pt an important distinction must be made. We have hitherto spoken of space as relational, & of spatial figures as relations. But space, it would seem, is rather relativity than relation — itself not a rel., it gives the bare possibility of rels. between distinct things. — — — — — — —

matter

But a bare ~~space~~ possibility cannot exist, or be given in sense percept. 99

Ewing

— — — It is not empty space, but spat. figures, which sense-percept reveals, & spatial figures involve a differentiation of space, & therefore a reference to the matter which is in space. It is spatial figures, also, & not empty space, with which G. has to deal. The antinomy discussed above arises, so it would seem, from the attempt to deal with empty space, rather than with spat. figures & the matter to which they necessarily refer.

Let us see whether, by this change, we can overcome the antinomy of the pt. Spatial figures, as we shall now say, are rels. between the matter which differentiates empty space. Their divisibility, which seemed to contradict their relational character, may be explained in two ways. First, as holding of the figures considered as parts of empty space, which is itself not a rel.; second, as denoting the possibility of continuous change in the rel.

expressed by the spatial figure. These two ways are, at bottom, the same; for empty space is a possibility of rels; & the figure, when viewed in connect with empty space, thus becomes a possible rel. with which other possible rels may be contrasted or compared. But the 2nd way of regarding divisibility is the better way, since it introduces a reference to the matter which differentiates empty space, without which, spatial figures, & therefore Y, could not exist. It is empty space, then, we must conclude which gives rise to the contrary question; for empty space is a bare possibility of rels, undifferentiated & homogeneous, & thus wholly destitute of parts or of things found. To speak of parts of a piss is nonsense; the parts & differentiation arise only through a reference to the matter which is differentiated in space.

But what nature must we ascribe to this matter, which is to be involved in all geom. propositions? In criticising Helmholtz, it may be remembered, we decided that Y refers to a peculiar & abstract kind of matter, which is not regarded as possessing any & causal qualities, as exerting or as subject to the action of forces. And this is the matter, I think, which we require for the needs of the moment. Not that we affirm, of course, that actual matter can be destitute of the properties with which Physics is cognizant, but that we abstract from those properties, as being irrelevant to Y. All that we require, for our immediate purpose, is a subject of that diversity which space renders possible, or terms for those rels by which empty space, if space is to be studied at all, must be differentiated. But how must a matter, which is to fulfil this function be regarded?

Empty space, as a possibility of diversity in rel, but spatial figures

191

with which G. necessarily deals, are the actual rel^s. rendered possible by empty space. Our matter, therefore, must supply the terms for these rels. It must be differentiated, since such diff. is the special work of space. We must find, therefore, in our matter, that unit of diff. or atom, which in space we could not find. This atom must be simple, i.e. it must contain no real diversity; it must be a This not resolvable into Thises. Being simple, it can contain no rels within itself, & consequently, since spatial figures are mere rels., it cannot appear as a spatial figure; for every spat. figure involves some diversity of matter. But our atom must have spat. rels. with other atoms, since to supply terms for these rels is its only funct. It is also capable of having these rels. since it is differentiated from other atoms. Hence we obtain an unextended term

[(2) The circle in the definit. of straight lines & planes. - - - Straight lines, in the purely geometrical [197] procedure, are defined only by pts, & pts only by straight lines. Such pts, are now replaced by material atoms: the duality of pts & lines, therefore, has disappeared, & the straight line may be defined as the spatial rel. between two unextended atoms. - - - Thus straight lines & planes are the true spatial units, & pts result only from the attempt to find, within space these]

for spat. rels, precisely of the kind we require. So long as we sought this term without reference to anything more than space, the self-contradictory not. of the pt. was the only outcome of our search; but now that we allow a ref. to the matter differentiated by space, we find at once the term which was needed, namely a non-spat. simple element with spat. rel^s. to other elements. To G. such a term will appear, owing to its spat. rels. as a pt; but the contradict^n of the pt. as we now see, is a result only of the undue abstract. with which G. deals. 192.

[terms for spatial rels which exist only in a more than spatial matter. Straight lines & planes & volumes are the spatial rels between 2, 3 or 4 unextended atoms & pts are a merely convenient geometrical fiction, by which possible atoms are replaced. For, in a space, as we saw, is a possibility, G. deals not with actually realized spatial rels but with the whole scheme of possible rels. 193.]

Antinomy pt = atom [illegible] (2) posit [illegible] defin [illegible] posit [illegible] defin [illegible] ~~relat indivisible~~ ~~spat. figure [illegible]~~ [illegible] Antinomy

(3) Space is at once relational & more than relational. [illegible] Antinomy [illegible]

Now for this discussion it is essential to distinguish clearly between empty

Spatial order

space & spat. figures. Empty sp. as a form of ediviuality, is not actual rels. but the possibility of rels. if we ascribe existential import to it, as the ground, in reality, of all diversity in rel, we at once have space as something not itself relative, though giving the possibility of all rels. In this sense, space is to be distinguished from spat. order. Spat. order, it may be said, presupposes space, as that in which this order is possible. Thus Stumpf says (Ursprung d. Raumv. s. 15): "There is no order or rel. without a ~~possible~~ positive absolute content, underlying it, & making it possible to order anything in this manner. Why then should we otherwise distinguish one order from another? ... To distinguish different orders from one another, we must everywhere recognize a particular absolute content, in rel. to which the order takes place. And so space, too, is not a mere order, but just that by which the spatial orders, side-

1913

by-sideness (Nebeneinander), distinguishes itself from the rest." 1913

- - - ↓ (3), 4

Let us consider what this empty space is. (I speak of "empty" space without necessarily implying the absence of matter, but only to denote a space which is not a mere order of mat. things.) Stumpf regards it as given in sense; Kant, in the last two arguments of his metaph. deduct., argues that it is an intuition, not a concept, & must be known before spat. order becomes possible. I wish to maintain, on the contrary, that it is wholly conceptual; that space is given only as spat. order; that sp. rels. being given, appear as more than mere rels. & so become hypostatized; that when hypostatized, the whole collect. of them is regarded as contained in empty space; but this empty space itself, if it means more than the logical possibility of space-rels. is an unnecessary & self-contradictory as-

sumption 194

Leibniz —— Kant —— Watson

mere rels. ideal but only; obj. reality absol. sp.

If empty space be the pre-condition of spat. order, we cannot expect it to be connected with sp. rels. as genus with species. But empty space may nevertheless be a universal concept; it may be related to spatial order as the state to the citizens. These are not instances of the state, but are contained in it; they also, in a sense, presuppose it, for a man can only become a citizen by being related to other citizens in a state. 195

Whence arises the paradox that we cannot but regard space as having more or less thinghood, & as divisible ad infinitum? This must be explained, I think, as a psychological illusion, due to unavoidably arising from the fact that sp. rels. are immediately presented. They thus have a

peculiar psychical quality as immediate experiences, by which quality they can be distinguished from time-rels. or any other order in which things may be arranged. To Stumpf, whose problem is psychol. such a psychical quality would constitute an absolute underlying content, & would fully justify his theses; to us, however, whose problem is epistemological, it would not do so, but would leave the meaning of the spat. element in sense-percept. free from any implication 196 of an absolute (Cf. James Psych. II. 148ff.) or empty space. May we not, then, abandon empty space, & say: Spat. order consists of felt rels. & qua felt has for Psych. an existence not wholly resolvable into rels., & unavoidably *seeming* to be more than mere rels. But when we examine the information, as to space which we derive from sense percept, we find ourselves plunged in contradictions, as soon as we allow this

information consist of more than rels. This leaves spat. ords. alone in the field, & reduces empty space to a mere name for the logical possibility of spat. rels.

The apparent divisibility of the rels. which constitute spat. ords. then, may be explained in two ways, though these are at bottom equivalent. We may take the rel. as considered in connect with empty space, in which case it becomes more than a rel, but being falsely hypostatized, it appears as a complex thing, necessarily composed of elements, which elements, however, nowhere emerge until we analyse the pseudo-thing down to nothing, & arrive at the pb. In this sense, the divisibility of spat. rels is an unavoidable illusion. Or again, we may take the rel in connect with the material atoms it relates. In this case, other atoms may be imagined, differently localized by different spat. rels. If they are localized on the straight line joining two of the original atoms, this straight line appears divided by them. But the original rel is not really divided: all that has happened is, that two or more equivalent rels have replaced it, as two compounded rels of father & son may replace the equivalent rel of grandfather & grandson. That two ways of viewing the apparent divisibility are equivalent for empty space, is so far as it is not illusion, is a name for the aggregate of possible spat. rels. To regard a figure in empty space as divided, therefore, means, if it means anything, to regard two or more other possible rels as substituted for it, which gives the second way of viewing the matter.

The same reference to matter, then, by which the antinomy of the Point was solved, solves also the antinomy as to the relational nature of space. Space, if it is to be freed from contradiction, must be regarded exclusively as spatial ords, as rels between unextended material atoms. Empty space,

which arises, by an inevitable illusion, out of the spatial element in sense-percept, may be regarded, if we wish to retain it, as the principle of relativity, the bare log. possibility of rels. between diverse things. In this sense, empty space is wholly conceptual; spat. order alone is immediately experienced.

But in what sense does spat. order consist of rels.? We have hitherto spoken of externality as a rel, & in a sense such a manner of speaking is justified. Externality, when predicated of anything, is an adjective of that thing, & implies a reference to some other thing. So far as this extent, then, externality is analogous to other rels.; & only to this extent, in our previous arguments, has it been regarded as a rel. But when we take account of further qualities of rels., externality begins to appear, not so much as a rel, but rather as a necessary aspect & element in every rel. And this is become out of by the necessity for the existence of rels. of some given form of externality.

Every rel, we may say, involves a diversity between the related terms, but also some unity. Mere diversity does not give a ground for that intercourse & that interdependence, which a rel requires. Mere unity leaves the terms identical, & thus destroys the reference of one to another required for a rel. Mere externality, taken in abstract, gives only the element of diversity required for a rel, & is thus more abstract than any actual rel. But mere diversity does not give that individible whole of which any actual rel must consist, & is thus, when regarded abstractly, not subject to the restrictions of ordinary rels.

But with mere diversity, we seem to have returned to empty space, & abandoned spat. order. Mere div., surely, is either complete or non-existent; degrees of div., or a quant. measure

of it, are nonsense. We cannot, therefore, reduce spat. ord. to mere diversity. Two things, if they occupy different positions in space, are necessarily diverse; but are as necessarily more; otherwise spat. order becomes unmeaning. 198

Empty space, then, in the above sense of the possibility of spatial rel's, contains only one aspect of a rel, namely the aspect of div.; but spat. order, by its reference to matter, becomes more concrete, & contains also the element of unity, arising out of the connection of the different material atoms. Spat. order, then, consists of rels in the ordinary sense; its merely spat. element, however, — if one may make such a distinction — the element, that is, which can be abstracted from matter & regarded as constituting empty space, is only one aspect of a rel, but an aspect which, in the concrete, must be inseparably bound up with the other aspect. Here, once more, we see the ground of the contradictions in empty space, & the reason why spat. order is free from these contradictions. 199

Couturat, Les Principes des Mathématiques

[illegible] n dimensions [illegible] ω dimensions space [illegible] point [illegible] Geom. [illegible] Topologie [illegible] 128–32

[illegible] biuniforme correspondence [illegible] continuité [illegible] 132–4

[illegible] continu [illegible] cardinal no. [illegible] equivalent, indiscernible [illegible] ordinal property [illegible] 133–4

Par conséquent, ce qui constitue proprement et essentiellement les continus à plusieurs dimensions (comme le continu linéaire), ce n'est pas un ensemble de points, mais un ensemble de relations. Ce fait a une portée philosophique manifeste; il signifie, en somme, que l'espace

n'est pas une simple « multiplicité », mais bien une multiplicité ordonnée, et il justifie la conception de Leibniz qui voyait dans l'espace, avant tout, un ordre. 134

„Bei der Untersuchung der Funktionen, welche aus der Integration vollständiger Differentiale entstehen, sind einige der Analysis situs angehörige Sätze fast unentbehrlich. Mit diesem von Leibniz, wenn auch vielleicht nicht ganz in derselben Bedeutung, gebrauchten Name darf wohl ein Teil der Lehre von den stetigen Grössen bezeichnet werden, welcher die Grössen nicht als unabhängig von der Lage existirend u. durch einander messbar betrachtet, sondern von den Massverhältnisse ganz absehend, nur ihre Orts- und Gebietsverhältnisse der Untersuchung unterwirft." (Riemann, Theorie der Abelschen Funktionen, §. 2. Journ. de Crelle. LIV)

Enriques. La Topologie, (Analysis Situs) ? théorie des connexions 1.78 –

Enriques. 12 –
L'objet essentiel de la Topologie, ce sont les connexions des surfaces et plus généralement, les relations de contiguïté et de continuité entre les points des lignes et des surfaces, abstraction faite de la grandeur de ces figures et même de leur forme. Plus précisément, les connexions sont les invariants des transformations biuniformes et continues, de sorte que ces transformations sont les seules qu'admette la Topologie. 140.

Proj. G. – 2 : point, ~~ligne~~ droite (element, classe) : notion indefinissable + 1 Ax pt ; term 2 : droite ; 4 = 3 : symétrique transitive : rel. : defin 2.4. Elle (G. proj.) n'implique non plus aucune proposition première indémontrable, puisque tous les postulats de la G. pr. font maintenant partie de la définition de l'espace projectif, et constituent les propriétés hypothétiques de cet espace. On n'en affirme aucune catégoriquement ; on se borne à affirmer que, *si* un espace jouit de telles propriétés énoncées dans sa définition, il possédera en

outre telles autres propriétés énoncées 158
dans les théorèmes. Ainsi la G. pr. est
ramenée à la forme d'une vaste
implicat., et par suite doit rentrer
dans la Math. pure, qui ne connaît
pas d'autres principes que ceux de
la Logique. 159

Congruence : 7 & 11 — superposit. ;
12 7 2 — Hystérim prolém +, Congruence + —
12 — superposable +, 20 : Congruence : not.
première +. 188

Comme la G. pr. et la G. des. la G. métrique peut se ramener à une forme purement logique si l'on transforme les postulats en une définition de l'espace métrique: Un espace métr. (euclidien ou non-eucl.) sera un ensemble qui jouira de telles et telles propriétés (énoncées dans les postulats). La G. mét. ou plutôt, chacune des G.M., prend alors la forme d'une vaste implicat.: Si tel ensemble jouit des propriétés fondamentales énoncées dans les postulats, il vérifiera tous les théorèmes de la G. correspondante. La G. ou plutôt les Gs. ne reposent plus sur des propositions premières, indémontrables; elles n'ont plus d'axiomes propres, en dehors des axiomes communs de la Logique même. 204

Une G. pure est une implicat. de la forme « Si A est vrai B est vrai »; la G. appliquée dit: « A est vrai, donc B est vrai »; elle affirme à la fois A et B de l'espace actuel, objectif, tandis que la G. pure n'affirme que la connexion logique, idéale, de A et de B. Or, entre toutes les G. logiquement possibles que l'on peut constituer théoriquement, l'expérience seule peut nous permettre de choisir celle que nous appliquerons au monde « réel », c'est-à-dire, au monde de notre expérience. Cela ne veut pas dire que, comme le croient l'empirisme et le réalisme naïf, il y ait hors de nous un espace tout fait

et réalisé que nous n'ayons qu'à percevoir; mais que le monde que nous percevons se prête plutôt à tel moule spatial qu'à tel autre. Cela ne veut pas dire non plus que l'on puisse vérifier par une expérience cruciale tel ou tel postulat isolé (par exemple le postulatum d'Euclide); car, comme on l'a fait remarquer, une telle vérificat. supposerait tous les autres postulats vérifiés, ce qui forme une sorte de cercle. Mais il ne faut pas non plus soutenir, comme font les agnosticistes contemporains, que toute vérificat. expérimentale de la G. constitue un cercle vicieux; car, si l'on ne peut pas vérifier chaque postulat séparément, on peut vérifier l'ensemble total des postulats. Seulement une telle vérificat. ne peut plus être directe et péremptoire: elle sera du genre de celles qui vérifient une hypothèse par ses con-

séquences, et par suite elle ne sera jamais que probable. Mais c'est là le cas de la plupart des hypothèses physiques; et cela ne fait que rapprocher la G. des sciences expérimentales. A ce point de vue, les postulats ne sont plus de simples « hypothèses » dans une implicat., ils deviennent des proposit. assertoriques, des vérités d'expérience. Il n'y a, on le voit, aucune inconséquence à considérer la G. pure comme une science a priori, et la G. appliquée 209 comme une science empirique; car il ne s'agit pas de la même G., ou plutôt, si c'est la même G., la modalité des proposit. y est toute différente. 210

G. appliquée [illegible] logical system [illegible] science [illegible] G. pure [illegible] 210.

[illegible] G. pure [illegible] Arith. [illegible] point [illegible] élément [illegible] ensemble [illegible] ensemble [illegible] 211

~~Il ne s'agit pas, à proprement parler, de construire l'espace avec des choses (ex pumice aquam !), mais seulement de construire avec des ens un ensemble qui ait toutes les propriétés fondamentales de l'espace considéré, afin de démontrer l'« existence » de cet espace sans faire appel à l'intuition. Entre des divers espaces que étudie la G. et les ensembles de ens que on leur substitue, il y a analogie formelle, et non identité ; mais cela suffit pour que on puisse démontrer l'existence (logique) de ces espaces sans aucun postulat intuitif ou expérimental, et par suite faire rentrer les divers G.s dans la Mathématique pure, commune dépendant uniquement de la Logique.~~ 213

En résumé, les axiomes de la G. ne sont que des définitions déguisées, ou plutôt des parties de définition. Mais alors (il importe de remarquer cette conséquence nécessaire de la conception que nous exposons ici), la G. ne peut pas être une science autonome ayant ses principes spéciaux et reposant sur des « jugements synthétiques a priori » ; c'est une série de déductions formelles suspendues à une définition, et qui en déroulent à l'infini les conséquences logiques. En un mot, la G. n'est plus qu'une simple promotion de la Logique. 205

De même que la G. n'a plus de propositions premières, elle n'a plus de notions premières qui lui soient propres. En effet, dans tous les Systèmes que nous avons exposés plus haut, les notions premières se réduisent à deux : un concept de classe que on nomme point, et une notion de relation (ordre, congruence, mouvement), qui se déguise parfois, quand l'analyse n'est pas poussée à bout, en un (Poincaré)

concept de classe (droite, segment, vecteur). Or, d'un côté, la notion de point n'intervient nullement dans la structure logique de la G : les points ne sont rien de plus que les éléments de certains ensembles, ou mieux, les termes de certaines relations ; ce sont des objets quelconques, de nature inconnue ou indéterminée, dont on ne sait qu'une chose : c'est qu'ils sont les supports de certaines relations. Ou plutôt, on n'en sait rien, même pas cela ; on sait seulement que, si des objets quelconques (qu'on les appelle points ou autrement) supportent entre eux certaines relations fondamentales (énoncées dans les axiomes), ils vérifieront toutes les théorèmes qui en découlent logiquement. Ainsi la G pure, qui n'est qu'un système d'implications, n'affirme rien touchant les points, rigoureusement parlant : elle ne les connaît pas, et elle n'en a pas besoin.

D'un autre côté, les relations qui constituent l'autre donnée primordiale des divers systèmes de G ne sont plus indéfinissables. Nous avons vu comment la Log. des rels permet de les définir au moyen de leurs propriétés formelles. 206.

C'est ce donc, en définitive, que la G., et comment doit-on la définir pour la distinguer des autres branches de la Math. ou de la Log. ? C'est « l'étude des suites à plusieurs dimensions » (Russel. R. of m. p. 372) « des suites », c'est-à-dire des ensembles ordonnées, « à plusieurs dimensions », car l'Arithm. par exemple, étudie une suite à une dimension, la suite naturelle des nos.

208

Pasch, Vorlesungen üb. Neuere G.

Was in der G. nichts weiter als Beschreibung gegründet (Helmholtz), freilich auf Beobachtungen, welche sich unzählich wiederholt haben, welche klaren erfasst werden, als die eines [illegible]

anderen Art, & mit denen die Menschen deshalb längst so vertraut geworden sind, dass ihr Ursprung in Vergessenheit gerathen u. Gegenstand des Streites werden konnte.

17.

Die Grundsätze sollen das von der Kultur zu verarbeitende empirische Material vollständig umfassen, so dass man auch ihre Aufstellung auf die Sinneswahrnehmungen nicht mehr zurückzugehen braucht.

17.

Allemal aber werden diejenige Körper, deren Zeichnung sich mit der Beobachtungsgrenze nicht verträgt, Punkte genannt.

5.